HOMOLOGICAL SYSTEMS IN TRIANGULATED CATEGORIES

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Abstract. We introduce the notion of homological systems Θ for triangulated categories. Homological systems generalize, on one hand, the notion of stratifying systems in module categories, and on the other hand, the notion of exceptional sequences in triangulated categories. We prove that, attached to the homological system Θ, there are two standardly stratified algebras A and B, which are derived equivalent. Furthermore, it is proved that the category $\mathfrak{F}(\Theta)$, of the Θ-filtered objects in a triangulated category $\mathcal{T}$, admits in a very natural way an structure of an exact category, and then there are exact equivalences between the exact category $\mathfrak{F}(\Theta)$ and the exact categories of the Δ-good modules associated to the standardly stratified algebras A and B. Some of the obtained results can be seen also under the light of the cotorsion pairs in the sense of Iyama-Nakaoka-Yoshino (see 6.6 and 6.7). We recall that cotorsion pairs are studied extensively in relation with cluster tilting categories, t-structures and co-t-structures.

1. Introduction.

In [1, 13, 14] were introduced the notion of standardly stratified algebras, generalizing the class of quasi-hereditary algebras. The standardly stratified algebras have shown to be homologically interesting because of their relationship with tilting theory and relative homological algebra. In order to give a categorification of the standard modules and the characteristic tilting module associated with an standardly stratified algebra, Erdmann and Sáenz developed the notion of Ext-injective stratifying system [16]. In that paper, they generalized the standard modules and the characteristic tilting module, obtained by Ringel in [30].

Afterwards, Marcos, Mendoza and Sáenz introduced the notions of stratifying system and Ext-projective stratifying system, and furthermore, they proved that all this notions are equivalent to the one given by Erdmann and Sáenz [23, 24]. In [24], they were able to prove that, for a given a stratifying system $(\Theta, \leq)$ in $\mathrm{mod}(A)$, there exists a module Q such that $B := \mathrm{End}(Q)^{op}$ is a standardly stratified algebra, and moreover, there exists an exact equivalence between the Θ-filtered modules in $\mathrm{mod}(A)$ and the Δ-good modules in $\mathrm{mod}(B)$. We remark that the considered order $\leq$, attached to a stratifying

system, is a finite linear one. On the other hand, Mendoza, Sáenz and Xi developed the theory of stratifying systems for the more general case of a finite pre-ordered set [25].

Triangulated categories have its origin in algebraic geometry and algebraic topology. This kind of categories have become relevant in many different areas of mathematics. Although the axioms of a triangulated category seems to be hard at first sight, it turns out that many categories are endowed with the structure of a triangulated category.

In this paper, we develop the concept of homological systems in the setting of artin triangulated R-categories. Throughout this notes, $\mathcal{T}$ will denote an arbitrary triangulated category and $[1] : \mathcal{T} \to \mathcal{T}$ its suspension functor. We introduce the notion of a Θ-system (see 5.1) in a triangulated category $\mathcal{T}$, which is the corresponding generalization of stratifying systems in the category of modules over an algebra A. We also state the concept of a Θ-projective system, and we show that a Θ-system determines a unique Θ-projective system (see 5.9). One of the main results of this paper, Theorem 7.4, says that for a given Θ-system in an artin triangulated R-category, there exist two standardly stratified algebras A and B; and moreover, we have triangulated equivalences $\mathrm{D}^b(\mathfrak{F}(\Theta)) \simeq \mathrm{D}^b(A)$ and $\mathrm{D}^b(\mathfrak{F}(\Theta)) \simeq \mathrm{D}^b(B)$. Furthermore, it is proved the Theorem 4.10, which is a generalization to the setting of triangulated categories of a well-known result obtained by Ringel [30, Theorem 1]. The Theorem 4.10 states that, for a given a family of objects $\Theta = \{\Theta(i)\}_i^n$ belonging to an artin triangulated R-category $\mathcal{T}$ and satisfying that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ if $j \geq i$, the subcategory $\mathfrak{F}(\Theta)$ of the Θ-filtered objects in $\mathcal{T}$ is functorially finite.

The notion of cotorsion pair in a triangulated category was recently introduced by Iyama-Yoshino [17] and Nakaoka [26]. This notion seems to be important since it unifies the notions of: (a) t-structures [9], (b) co-t-structures [28] and (c) cluster tilting subcategories [20]. By using the theory of homological systems, it is constructed two canonical cotorsion pairs (see 6.6) and it is also determined the core of those cotorsion pairs.

As a beautiful application (see Theorem 8.3), of the theory of homological systems, we showed that if $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ is a strongly exceptional sequence in the triangulated category $\mathcal{T}$, then there exists an equivalence $\mathrm{D}^b(\mathfrak{F}(\mathcal{E})) \simeq \mathrm{D}^b(A)$ as triangulated categories for some quasi-hereditary algebra A. Observe that this result generalize [7, Theorem 6.2].
A similar result (see Theorem 8.5) holds for an exceptional sequence $\mathcal{E}$ in the bounded derived category $\mathrm{D}^b(\mathcal{H})$, where $\mathcal{H}$ is a hereditary abelian k-category. We recall that the notion of exceptional sequences has its origin from the study of vector bundles on projective spaces (see, for instance, [7, 19]) and that strongly exceptional sequences appear very often in algebraic geometry and provides a non-commutative model for the study of algebraic varieties (see [7]).

The paper is organized as follows: In section 2, we give some basic notions and properties of triangulated categories, which will be used in the rest of the work.
In section 3, it is established the concept of artin triangulated R-category and we give some technical results that will be useful when proving that a Θ-system determines a unique Θ-projective system.
In section 4, we study the subcategory $\mathfrak{F}(\Theta)$ of Θ-filtered objects in a triangulated category $\mathcal{T}$ and it is proved the Theorem 4.10, which is a generalization of [30, Theorem 1].
In section 5, we focus our attention into the Θ-systems. It is shown that a Θ-system determines a unique Θ-projective system. We also prove that, for a given Θ-projective system, the filtration multiplicity $[M : \Theta(i)]_\xi$ does not depend on the given filtration ξ.
In section 6, we show that, for a given Θ-projective system $(\Theta, \mathbf{Q}, \leq)$ in $\mathcal{T}$, there exists an equivalence between $\mathfrak{F}(\Theta)$ and the subcategory of the Δ-good modules in $\mathrm{mod}(B)$, for some standardly stratified algebra B.
In Section 7, we show that the triangulation in $\mathcal{T}$ induces in a natural way an exact structure in $\mathfrak{F}(\Theta)$, and prove Theorem 7.4, which is one of the main results of the paper.
Finally, in section 8, we provide some examples of homological systems.

2. Preliminaries

In this paper, $\mathcal{T}$ will be a triangulated category and $[1] : \mathcal{T} \to \mathcal{T}$ its suspension (shift) functor. Moreover, when we say that $\mathcal{C}$ is a subcategory of $\mathcal{T}$, it always means that $\mathcal{C}$ is a full subcategory which is additive and closed under isomorphisms. For a class $\mathcal{X}$ of objects of $\mathcal{T}$, we denote by $\mathrm{add}\,(\mathcal{X})$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$, closed under finite direct sums and direct summands.

For some classes $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{T}$, we write ${}^{\perp}\mathcal{X} := \{Z \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(Z, -)|_{\mathcal{X}} = 0\}$ and $\mathcal{X}^{\perp} := \{Z \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(-, Z)|_{\mathcal{X}} = 0\}$. We also recall that $\mathcal{X} * \mathcal{Y}$ denotes the class of objects $Z \in \mathcal{T}$ for which there exists a distinguished triangle $X \to Z \to Y \to X[1]$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Furthermore, it is said that $\mathcal{X}$ is **closed under extensions** if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.

We will make use of the following constructions in triangulated categories: the base and co-base change. These constructions remind us the pull-back and the push-out, respectively, of short exact sequences in abelian categories.

Proposition 2.1. [6, 2.1] *For any triangulated category $\mathcal{T}$, each one of the following conditions is equivalent to the octahedral axiom.*

(a) *BASE CHANGE. For any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and any morphism $\epsilon : E \to C$ in $\mathcal{T}$, there exists a commutative*

diagram in $\mathcal{T}$

$$\begin{tikzcd}
 & M \arrow[r, equal] \arrow[d, "\alpha"] & M \arrow[d, "\delta"] & \\
A \arrow[r, "f'"] \arrow[d, equal] & G \arrow[r, "g'"] \arrow[d, "\beta"] & E \arrow[r, "h'"] \arrow[d, "\epsilon"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\gamma"] & C \arrow[r, "h"] \arrow[d, "\zeta"] & A[1] \\
 & M[1] \arrow[r, equal] & M[1], &
\end{tikzcd}$$

where all the rows and columns, in the preceding diagram, are distinguished triangles.

(b) *CO-BASE CHANGE. For any distinguished triangle* $A \stackrel{f}{\to} B \stackrel{g}{\to} C \stackrel{h}{\to} A[1]$ *and any morphism* $\alpha : A \to D$ *in* $\mathcal{T}$, *there exists a commutative diagram in* $\mathcal{T}$

$$\begin{tikzcd}
 & N \arrow[r, equal] \arrow[d, "\zeta"] & M \arrow[d, "\delta"] & \\
C[-1] \arrow[r, "-h[-1]"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, equal] \\
C[-1] \arrow[r, "-h'[-1]"] & D \arrow[r, "f'"] \arrow[d, "\eta"] & F \arrow[r, "g'"] \arrow[d, "\vartheta"] & C \\
 & N[1] \arrow[r, equal] & N[1], &
\end{tikzcd}$$

where all the rows and columns, in the preceding diagram, are distinguished triangles.

Lemma 2.2. *Consider the following commutative diagram in a triangulated category* $\mathcal{T}$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d, "\beta'"] & C \arrow[r] & A[1] \\
A' \arrow[r, "\alpha'"] & B' \arrow[r] & C' \arrow[r] & A'[1],
\end{tikzcd}$$

where the rows are distinguished triangles. Then, the preceding diagram can be completed to the following one

$$\begin{tikzcd}
A''[-1] \arrow[r] \arrow[d] & B''[-1] \arrow[r] \arrow[d] & C''[-1] \arrow[r] \arrow[d] & A'' \arrow[d] \\
A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d, "\beta'"] & C \arrow[r] \arrow[d, "\Phi"] & A[1] \arrow[d] \\
A' \arrow[r, "\alpha'"] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] \arrow[dr, phantom, "IX"] & A'[1] \arrow[d] \\
A'' \arrow[r] & B'' \arrow[r] & C'' \arrow[r] & A''[1],
\end{tikzcd}$$

where the rows and columns, in the above diagram, are distinguished triangles and all the squares commute, except by the one marked with IX, which anti-commutes.

Proof. By completing β and β' to distinguished triangles, we have the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"] & A' \arrow[r, "\gamma"] \arrow[d, "\alpha'"] & A'' \arrow[r, "\delta"] & A[1] \\
B \arrow[r, "\beta'"] & B' \arrow[r, "\gamma'"] & B'' \arrow[r, "\delta'"] & B[1].
\end{tikzcd}$$

Then, there is a morphism $h : A'' \to B''$ in $\mathcal{T}$, such that the triple (α, α', h) is a morphism of triangles. Hence $h[-1] : A''[-1] \to B''[-1]$ makes commutative the following square

$$\begin{tikzcd}
A''[-1] \arrow[r, "h[-1]"] \arrow[d, "-\delta[-1]"'] & B''[-1] \arrow[d, "-\delta'[-1]"] \\
A \arrow[r, "\alpha"] & B.
\end{tikzcd}$$

Then, by a Verdier's result (see Exercise 10.2.6, page 378, in [33]), we get the lemma. $\square$

The following result remind us the so-called Snake's Lemma.

Proposition 2.3. *Consider the following commutative diagram in a triangulated category* $\mathcal{T}$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d, "\beta'"] & C \arrow[r] \arrow[d, "\beta''"] & A[1] \arrow[d, "\beta[1]"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r] & C' \arrow[r] & A'[1], \end{tikzcd}$$

where the rows are distinguished triangles. If $\mathrm{Hom}_{\mathcal{T}}(A, C'[-1]) = 0$ *then the preceding diagram can be completed to the following one*

$$\begin{tikzcd} A''[-1] \arrow[r] \arrow[d] & B''[-1] \arrow[r] \arrow[d] & C''[-1] \arrow[r] \arrow[d] & A'' \arrow[d] \\ A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d, "\beta'"] & C \arrow[r] \arrow[d, "\beta''"] & A[1] \arrow[d, "\beta[1]"] \\ A' \arrow[r, "\alpha'"] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] \arrow[dr, phantom, "IX"] & A'[1] \arrow[d] \\ A'' \arrow[r] & B'' \arrow[r] & C'' \arrow[r] & A''[1], \end{tikzcd}$$

where the rows and columns, in the above diagram, are distinguished triangles and all the squares commute, except the one marked with IX, which anti-commutes.

Proof. Assume that $\mathrm{Hom}_{\mathcal{T}}(A, C'[-1]) = 0$. By 2.2, the square given (in the first diagram) by the morphism α and β can be completed to a diagram as in 2.2. We only need to prove that $\Phi = \beta''$, but this fact follows from [18, Corollary 5] page 243, since $\mathrm{Hom}_{\mathcal{T}}(A, C'[-1]) = 0$. $\square$

Definition 2.4. *Let* $\mathcal{T}$ *be a triangulated category and* $\mathcal{A}$ *be an abelian category. Consider subcategories* $\mathcal{X} \subseteq \mathcal{T}$ *and* $\mathcal{W} \subseteq \mathcal{A}$, *which are both closed under extensions. It is said that:*

(a) *a distinguished triangle* $\eta :\ A \to B \to C \to A[1]$ *belongs to* $\mathcal{X}$, *that is* $\eta \in \mathcal{X}$, *if the objects* A, B *and* C *belong to* $\mathcal{X}$;
(b) *an additive functor* $F : \mathcal{X} \to \mathcal{W}$ *is* ***exact****, if for every distinguished triangle* $\eta :\ A \to B \to C \to A[1]$ *in* $\mathcal{X}$, *we have that the sequence* $F(\eta) :\ 0 \to F(A) \to F(B) \to F(C) \to 0$ *is exact and belongs to* $\mathcal{W}$.

We also recall the following well-known definition (see, for example, [8] and [10]).

Definition 2.5. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be classes of objects in a triangulated category* $\mathcal{T}$. *A morphism* $f : X \to C$ *in* $\mathcal{T}$ *is said to be an* $\mathcal{X}$**-precover** *of* C *if* $X \in \mathcal{X}$ *and* $\mathrm{Hom}_{\mathcal{T}}(X', f) : \mathrm{Hom}_{\mathcal{T}}(X', X) \to \mathrm{Hom}_{\mathcal{T}}(X', C)$ *is surjective* $\forall X' \in \mathcal{X}$. *If*

any $C \in \mathcal{Y}$ admits an $\mathcal{X}$-precover, then $\mathcal{X}$ is called a **precovering** *class in $\mathcal{Y}$. By dualizing the definition above, we get the notion of an* **$\mathcal{X}$-preenvelope** *of C and a* **preenveloping** *class in $\mathcal{Y}$. Finally, it is said that $\mathcal{X}$ is* **functorially finite** *in $\mathcal{T}$ if $\mathcal{X}$ is both precovering and preenveloping in $\mathcal{T}$.*

In what follows, we recall some notions and elementary well-known facts about standardly stratified algebras. Let Λ be an artin R-algebra. We denote by $\mathrm{mod}\,(\Lambda)$ to the category of all finitely generated left Λ-modules, and by $\mathrm{proj}\,(\Lambda)$ to the full subcategory of $\mathrm{mod}\,(\Lambda)$ whose objects are the projective Λ-modules. For $M, N \in \mathrm{mod}\,(\Lambda)$, the **trace** $\mathrm{Tr}_M\,(N)$ of M in N, is the Λ-submodule of N generated by the images of all morphisms from M to N. For a given natural number t, we set $[1,t] = \{1, 2, \cdots, t\}$.

We next recall the definition (see [1, 14, 15, 30]) of the class of standard Λ-modules. Let n be the rank of the Grothendieck group $K_0\,(\Lambda)$. We fix a linear order $\leq$ on the set $[1,n]$ and a representative set ${}_\Lambda P = \{{}_\Lambda P(i) \ : \ i \in [1,n]\}$ containing one module of each iso-class of indecomposable projective Λ-modules. The set of **standard Λ-modules** is ${}_\Lambda\Delta = \{{}_\Lambda\Delta(i) : i \in [1,n]\}$, where ${}_\Lambda\Delta(i) = {}_\Lambda P(i)/\mathrm{Tr}_{\oplus_{j>i}\,{}_\Lambda P(j)}\,({}_\Lambda P(i))$. Then, ${}_\Lambda\Delta(i)$ is the largest factor module of ${}_\Lambda P(i)$ with composition factors only amongst ${}_\Lambda S(j)$ for $j \leq i$.

Let $\mathfrak{F}({}_\Lambda\Delta)$ be the subcategory of $\mathrm{mod}\,(\Lambda)$ consisting of the Λ-modules having a ${}_\Lambda\Delta$-filtration, that is, a sequence of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ with factors M_{i+1}/M_i isomorphic to a module in ${}_\Lambda\Delta$ for all i. The pair $(\Lambda, \leq)$ is said to be a **standardly stratified algebra**, with respect to the linear order $\leq$ on the set $[1,n]$, if $\mathrm{proj}\,(\Lambda) \subseteq \mathfrak{F}({}_\Lambda\Delta)$ (see [1, 13, 14]). A **quasi-hereditary algebra** is a standardly stratified algebra $(\Lambda, \leq)$ such that $\mathrm{End}({}_\Lambda\Delta(i))$ is a division ring, for each $i \in [1,n]$.

3. TRIANGULATED R-CATEGORIES

Let R be a commutative ring. We recall that an **R-category** is a category $\mathcal{C}$ satisfying the following two conditions: (a) for each pair X, Y of objects in $\mathcal{C}$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an R-module, and (b) the composition of morphisms in $\mathcal{C}$ is R-bilinear. An R-category $\mathcal{C}$ is called **Hom-finite** if $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finitely generated R-module, for each $X, Y \in \mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$, between R-categories, is said to be an **R-functor** if $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a morphism of R-modules for each pair X, Y of objects in $\mathcal{C}$.

Definition 3.1. *A* **triangulated R-category** *is an R-category $\mathcal{T}$ which is a triangulated one, and such that its shift functor $[1] : \mathcal{T} \to \mathcal{T}$ is an R-functor.*

Let $\mathcal{C}$ be an additive category. It is said that $\mathcal{C}$ is **Krull-Schmidt** if any object $X \in \mathcal{T}$ has a finite decomposition $X = \oplus_{i=1}^n X_i$ satisfying that each X_i is indecomposable with local endomorphism ring $\mathrm{End}_{\mathcal{C}}(X_i)$. An idempotent

$e = e^2 \in \mathrm{End}_{\mathcal{C}}(X)$ **splits** if there are morphism $u : X \to Y$ and $v : Y \to X$ satisfying $e = vu$ and $1_Y = uv$.

The following result is well-known and a proof can be found, for example, in [11]: An additive category $\mathcal{C}$ is Krull-Schmidt if and only if any idempotent in $\mathcal{C}$ splits and $\mathrm{End}_{\mathcal{C}}(X)$ is a semi-perfect ring for any $X \in \mathcal{C}$. In this case, any object $X \in \mathcal{C}$ has a unique (up to order) finite direct decomposition $X = \oplus_{i=1}^{n} X_i$ satisfying that each X_i is indecomposable with local endomorphism ring $\mathrm{End}_{\mathcal{C}}(X_i)$.

Definition 3.2. *A category $\mathcal{T}$ is said to be an* **artin triangulated R-category** *if the following conditions hold.*

(a) *$\mathcal{T}$ is a triangulated R-category, where R is an artinian ring.*
(b) *$\mathcal{T}$ is Hom-finite and Krull-Schmidt.*

Let Λ be an artin R-algebra. It is also well-known that the bounded derived category $D^b(\Lambda)$, of complexes in $\mathrm{mod}\,(\Lambda)$, is an artin triangulated R-category (see, for example, in [11, Theorem B.2]).

Proposition 3.3. *Let $\mathcal{T}$ be an artin triangulated R-category, $A \in \mathcal{T}$, $\Gamma := \mathrm{End}_{\mathcal{T}}(A)^{op}$ and the evaluation functor at A, $e_A := \mathrm{Hom}_{\mathcal{T}}(A, -) : \mathcal{T} \to \mathrm{Mod}\,(\Gamma)$. Then, the following conditions hold.*

(a) *Γ is an artin R-algebra.*
(b) *The restriction, $e_A : \mathcal{T} \to \mathrm{mod}\,(\Gamma)$, is well defined and induces an equivalence of categories $\mathrm{add}\,(A) \xrightarrow{\sim} \mathrm{proj}\,(\Gamma)$.*
(c) *$e_A : \mathrm{Hom}_{\mathcal{T}}(Z, X) \to \mathrm{Hom}_{\Gamma}(e_A(Z), e_A(X))$ is an isomorphism of R-modules for any $Z \in \mathrm{add}\,(A)$ and $X \in \mathcal{T}$.*

Proof. The proof done by M. Auslander (see [3]) can be easily extended to the context of an artin triangulated R-category. $\square$

Lemma 3.4. *Let $\mathcal{T}$ be a Hom-finite triangulated R-category, and let $A, C \in \mathcal{T}$ be such that $\mathrm{Hom}_{\mathcal{T}}(C[-1], A) \neq 0$. Then, the following conditions holds.*

(a) *There exists a not splitting distinguished triangle in $\mathcal{T}$*

$$\eta_{C,A} : \quad A^n \xrightarrow{f} E \xrightarrow{g} C \xrightarrow{h} A^n[1]$$

such that $\mathrm{Hom}_{\mathcal{T}}(-h[-1], A) : \mathrm{Hom}_{\mathcal{T}}(A^n, A) \to \mathrm{Hom}_{\mathcal{T}}(C[-1], A)$ is surjective, where $n := \ell_R(\mathrm{Hom}_{\mathcal{T}}(C[-1], A))$.
(b) *If $\mathrm{Hom}_{\mathcal{T}}(A, A[1]) = 0$ then $\mathrm{Hom}_{\mathcal{T}}(E, A[1]) = 0$.*

Proof. (a) Since $n := \ell_R(\mathrm{Hom}_{\mathcal{T}}(C[-1], A))$, it follows that there exists a family $\{h_i\}_{i=1}^{n}$ of R-generators in $\mathrm{Hom}_{\mathcal{T}}(C, A[1])$. Hence, for each $i \in [1, n]$, we have the corresponding distinguished triangle

$$\eta_i : \; A \xrightarrow{f_i} B_i \xrightarrow{g_i} C \xrightarrow{h_i} A[1] \, .$$

By taking $\xi := \oplus_{i=1}^n \eta_i$, we obtain the distinguished triangle $\xi : A^n \to \oplus_{i=1}^n B_i \to C^n \to A^n[1]$. Let $\Delta : C \longrightarrow C^n$ be the diagonal morphism. Then, by base change (see 2.1), we get the following commutative diagram

$$\begin{tikzcd}
\eta_{C,A}: A^n \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "K"] & C \arrow[r, "h"] \arrow[d, "\Delta"] & A^n[1] \arrow[d, equal] \\
A^n \arrow[r] & \oplus_{i=1}^n B_i \arrow[r] & C^n \arrow[r] & A^n[1],
\end{tikzcd}$$

where the rows are distinguished triangles. Consider now, the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
C[-1] \arrow[r, "-h[-1]"] \arrow[d, "\Delta[-1]"'] & A^n \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "K"] & C \arrow[r, "h"] \arrow[d, "\Delta"] & A^n[1] \arrow[d, equal] \\
C^n[-1] \arrow[r, "-\varphi[-1]"] \arrow[d, "\pi_i''[-1]"'] & A^n \arrow[r] \arrow[d, "\pi_i"] & \oplus_{i=1}^n B_i \arrow[r] \arrow[d, "\pi_i'"] & C^n \arrow[r, "\varphi"] \arrow[d, "\pi_i''"] & A^n[1] \arrow[d, "\pi_i[1]"] \\
C[-1] \arrow[r, "-h_i[-1]"] & A \arrow[r] & B_i \arrow[r] & C \arrow[r] & A[1],
\end{tikzcd}$$

where π_i, π_i' and π_i'' are the corresponding canonical projections of the direct sum. By the preceding diagram, we have that $\pi_i(-h[-1]) = -h_i[-1]$ for all $i \in [1, n]$ and since the shift $[1] : \mathcal{T} \to \mathcal{T}$ is an R-functor, we get that the set $\{h_i[-1]\}_{i=1}^n$ is an R-generator of $\mathrm{Hom}_{\mathcal{T}}(C[-1], A)$. Thus the map

$$\mathrm{Hom}_{\mathcal{T}}(-h[-1], A) : \mathrm{Hom}_{\mathcal{T}}(A^n, A) \to \mathrm{Hom}_{\mathcal{T}}(C[-1], A)$$

is surjective. Finally, by using the fact that $h_i \neq 0$ for each i, it follows that $h \neq 0$ and therefore the triangle $\eta_{C,A}$ does not split.

(b) Let $\mathrm{Hom}_{\mathcal{T}}(A, A[1]) = 0$. Applying $\mathrm{Hom}_{\mathcal{T}}(-, A)$ to the triangle $\eta_{C,A}$, from the item (a), we have the following exact sequence

$$(A^n, A) \xrightarrow{(-h[-1], A)} (C[-1], A) \longrightarrow (E[-1], A) \longrightarrow (A^n[-1], A)\ .$$

But, since $\mathrm{Hom}_{\mathcal{T}}(A^n[-1], A) = 0$ and the map $\mathrm{Hom}_{\mathcal{T}}(-h[-1], A)$ is surjective, it follows that $\mathrm{Hom}_{\mathcal{T}}(E[-1], A) = 0$. $\square$

Proposition 3.5. *Let $\mathcal{T}$ be an artin triangulated R-category and let $\eta : A \xrightarrow{s} B \xrightarrow{g} C \xrightarrow{\Psi} A[1]$ be a not splitting distinguished triangle such that $\mathrm{Hom}_{\mathcal{T}}(A, C) = \mathrm{Hom}_{\mathcal{T}}(A[1], C) = 0$ and C is an indecomposable object. Then, there exists a not splitting distinguished triangle $A' \to B' \to C \to A'[1]$ such that A' is a direct summand of A and B' is an indecomposable direct summand of B.*

Proof. Denote by $\alpha(B)$ the number of indecomposable direct summands that appear in a decomposition of B as direct sum of indecomposables. The

proof will be done by induction on $\alpha(B)$. If $\alpha(B)=1$, there is nothing to prove.

Let $\alpha(B)>1$. Consider a decomposition $B=B_1\oplus B_2$ with B_1 indecomposable. Then, the triangle η can be written as follows

$$\eta:\ A \xrightarrow{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}} B_1\oplus B_2 \xrightarrow{\begin{pmatrix} g_1 & g_2 \end{pmatrix}} C \longrightarrow A[1]\ .$$

Applying $\mathrm{Hom}_{\mathcal{T}}(-,C)$ to η, we have the following exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(A[1],C) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(C,C) \xrightarrow{(g,C)} \mathrm{Hom}_{\mathcal{T}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A,C)\ .$$

Since $\mathrm{Hom}_{\mathcal{T}}(A,C)=\mathrm{Hom}_{\mathcal{T}}(A[1],C)=0$, it follows that $\mathrm{Hom}_{\mathcal{T}}(g,C)$ is an isomorphism. Let us consider the morphisms $(g_1,0):B\to C$ and $(0,g_2):B\to C$. Therefore there exists $f,\ f':C\to C$ such that $f(g_1,g_2)=(g_1,0)$ and $f'(g_1,g_2)=(0,g_2)$. So, we get the following equalities

$$\begin{aligned} fg_1 &= g_1,\\ fg_2 &= 0,\\ f'g_1 &= 0,\\ f'g_2 &= g_2. \end{aligned}$$

Observe that $\mathrm{Hom}_{\mathcal{T}}(g,C)(f+f')=(g_1,g_2)$, and since $\mathrm{Hom}_{\mathcal{T}}(g,C)(1_C)=(g_1,g_2)$, we have that $f+f'=1_C$. We claim now that f and f' are idempotents. Indeed, we see, first, that $ff'=f'f=0$. The equality $ff'=0$ follows from the fact that $\mathrm{Hom}_{\mathcal{T}}(g,C)(ff')=(0,0)$ since $\mathrm{Hom}_{\mathcal{T}}(g,C)$ is an isomorphism, and similarly we also get that $f'f=0$.
Now, from the equality $f+f'=1_C$, we get that $f^2+ff'=f$ and then $f^2=f$. Analogously, it can be shown that $f'^2=f'$. Furthermore, since $\mathcal{T}$ is Krull-Schmidt and C is indecomposable, it follows that either $f=0$ or $f'=0$. Hence, by the equalities listed above, we get that either $g_1=0$ or $g_2=0$.

Assume that $g_1=0$. Consider the following distinguished triangles

$$C[-1] \xrightarrow{h_2} W' \xrightarrow{\delta'} B_2 \xrightarrow{g_2} C \quad \text{and} \quad 0 \longrightarrow B_1 \xrightarrow{1_{B_1}} B_1 \longrightarrow 0\ ,$$

where the first triangle is constructed by using the morphism g_2. Thus, by taking their direct sum, we get the following distinguished triangle

$$C[-1] \xrightarrow{\begin{pmatrix} 0 \\ h_2 \end{pmatrix}} B_1\oplus W' \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \delta' \end{pmatrix}} B_1\oplus B_2 \xrightarrow{\begin{pmatrix} 0 & g_2 \end{pmatrix}} C\ .$$

So, we can construct the following commutative diagram

$$\begin{tikzcd} C[-1] \arrow[r] \arrow[d, equal] & A \arrow[r, "\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}"] & B_1 \oplus B_2 \arrow[r, "\begin{pmatrix} 0 & g_2 \end{pmatrix}"] \arrow[d, equal] & C \arrow[d, equal] \\ C[-1] \arrow[r, "\begin{pmatrix} 0 \\ h_2 \end{pmatrix}"] & B_1 \oplus W' \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & \delta' \end{pmatrix}"] & B_1 \oplus B_2 \arrow[r, "\begin{pmatrix} 0 & g_2 \end{pmatrix}"] & C, \end{tikzcd}$$

where the rows are distinguished triangles. Hence, there exists an isomorphism $\xi : A \to B_1 \oplus W'$ inducing an isomorphism of triangles. In particular, W' is a direct summand of A. On the other hand, we have the following commutative diagram

$$\begin{tikzcd} W' \arrow[r, "\delta'"] & B_2 \arrow[r, "g_2"] \arrow[d] & C \arrow[r, "-h_2[1]"] \arrow[d, equal] & W'[1] \\ A \arrow[r, "s"] & B \arrow[r, "g"] & C \arrow[r, "\Psi"] & A[1], \end{tikzcd}$$

where the rows are distinguished triangles. From this diagram, we get a morphism $\beta' : W' \longrightarrow A$ inducing a morphism of triangles. Consider the following distinguished triangle

$$\eta' : \quad W' \xrightarrow{\delta'} B_2 \xrightarrow{g_2} C \xrightarrow{-h_2[1]} W'[1].$$

We assert that η' does not split. Indeed, if η' splits, we have that $-h_2[1] = 0$ and then $\Psi = \beta'[1](-h_2[1]) = 0$; thus the triangle η splits, which is a contradiction proving that η' does not split. Moreover $\mathrm{Hom}_{\mathcal{T}}(W', C) = \mathrm{Hom}_{\mathcal{T}}(W'[1], C) = 0$ since W' is a direct summand of A. We also have that $\alpha(\beta_2) < \alpha(\beta)$. Hence, by induction, the result follows. Finally, the case $g_2 = 0$ is analogous. This completes the proof. □

4. Filtered objects in a triangulated category

Let $\mathcal{X}$ be a class of objects in a triangulated category $\mathcal{T}$. It is said that an object $M \in \mathcal{T}$ admits an **$\mathcal{X}$-filtration** if there is a family of distinguished triangles $\eta = \{\eta_i \ : \ M_{i-1} \to M_i \to X_i \to M_{i-1}[1]\}_{i=0}^n$ such that $M_{-1} = 0 = X_0$, $M_n = M$ and $X_i \in \mathcal{X}$ for $i \geq 1$. In such a case, it is defined the lengths: $\ell_{\mathcal{X},\eta}(M) := n$ and $\ell_{\mathcal{X}}(M) := \min\{\ell_{\mathcal{X},\eta}(M) \mid \eta \text{ is an } \mathcal{X}\text{-filtration of } M\}$. Finally, it is denoted by $\mathfrak{F}(\mathcal{X})$ the class of objects $M \in \mathcal{T}$ for which there exists an $\mathcal{X}$-filtration.

Remark 4.1. *For a triangulated category $\mathcal{T}$ and a class $\mathcal{X}$ of objects in $\mathcal{T}$, the following statements hold.*

(a) $\mathfrak{F}(\mathcal{X}) = \cup_{n\in\mathbb{N}} \mathfrak{F}_n(\mathcal{X})$, *where* $\mathfrak{F}_0(\mathcal{X}) := \{0\}$ *and* $\mathfrak{F}_n(\mathcal{X}) := \mathfrak{F}_{n-1}(\mathcal{X}) * \mathcal{X}$ *for* $n \geq 1$.
(b) $\ell_{\mathcal{X}}(M) = \min\{n \in \mathbb{N} \mid M \in \mathfrak{F}_n(\mathcal{X})\}$ *for any* $M \in \mathfrak{F}(\mathcal{X})$.
(c) $\mathfrak{F}(\mathcal{X}[i]) = \mathfrak{F}(\mathcal{X})[i]$ *for all* $i \in \mathbb{Z}$. *Indeed, it can be seen that* $(\mathcal{X} * \mathcal{Y})[i] = (\mathcal{X}[i]) * (\mathcal{Y}[i])$ *for any classes* $\mathcal{X}$ *and* $\mathcal{Y}$ *of objects in* $\mathcal{T}$. *Hence, (c) follows from (a).*

Lemma 4.2. *Let* $\mathcal{X}$ *be a class of objects in a triangulated category* $\mathcal{T}$. *Then, the class* $\mathfrak{F}(\mathcal{X})$ *is closed under extensions.*

Proof. Let $A \to B \to C \to A[1]$ be a distinguished triangle in $\mathcal{T}$ with A and C in $\mathfrak{F}(\mathcal{X})$. The proof will be done by induction on $n := \ell_{\mathcal{X}}(C)$. If $C = 0$, we have that $A \simeq B$ and hence $B \in \mathfrak{F}(\mathcal{X})$.
If $\ell_{\mathcal{X}}(C) = 1$ then $C \simeq X \in \mathcal{X}$, and therefore an $\mathcal{X}$-filtration of B can be done by adding the triangle $A \to B \to C \to A[1]$ to an $\mathcal{X}$-filtration of A.

Suppose that $\ell_{\mathcal{X}}(C) > 1$. Consider a minimal $\mathcal{X}$-filtration of C,

$$\{\eta_i : \ C_{i-1} \longrightarrow C_i \longrightarrow X_i \longrightarrow C_{i-1}[1] \ \}_{i=0}^{n}.$$

By base change (see 2.1), we obtain the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
 & X_n[-1] \arrow[r, equal] \arrow[d] & X_n[-1] \arrow[d] & \\
A \arrow[r] \arrow[d, equal] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & A[1] \arrow[d, equal] \\
A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \\
 & X_n \arrow[r, equal] & X_n &
\end{tikzcd}$$

where the rows and columns are distinguished triangles and $\ell_{\mathcal{X}}(C_{n-1}) < \ell_{\mathcal{X}}(C)$. Applying induction to the first row of the preceding diagram, we get that $B_{n-1} \in \mathfrak{F}(\mathcal{X})$. Therefore, an $\mathcal{X}$-filtration of B is given by adding the triangle $B_{n-1} \to B \to X_n \to B_{n-1}[1]$ to an $\mathcal{X}$-filtration of B_{n-1}. $\square$

Lemma 4.3. *Let* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be classes of objects in a triangulated category* $\mathcal{T}$. *If* $\mathrm{Hom}_{\mathcal{T}}(\mathcal{Y}, \mathcal{Z}[i]) = 0$ *for some* $i \in \mathbb{Z}$, *then* $\mathrm{Hom}_{\mathcal{T}}(\mathfrak{F}(\mathcal{Y}), \mathfrak{F}(\mathcal{Z})[i]) = 0$.

Proof. Let $\mathrm{Hom}_{\mathcal{T}}(\mathcal{Y}, \mathcal{Z}[i]) = 0$ for some $i \in \mathbb{Z}$. By 4.1 (c), it is enough to prove the result only for the case $i = 0$. So, we assume that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{Y}, \mathcal{Z}) = 0$ and we prove that $\mathrm{Hom}_{\mathcal{T}}(\mathfrak{F}(\mathcal{Y}), \mathfrak{F}(\mathcal{Z})) = 0$.

Let $N \in \mathfrak{F}(\mathcal{Y})$ and $M \in \mathfrak{F}(\mathcal{Z})$. We will show, by induction on $\ell_{\mathcal{Y}}(N)$, that $\mathrm{Hom}_{\mathcal{T}}(N, M) = 0$. In order to do that, we also can assume that $M \neq 0$ and $N \neq 0$.

If $\ell_{\mathcal{Y}}(N)=1$ then $N\simeq Y\in\mathcal{Y}$ and so, by induction on $\ell_{\mathcal{Z}}(M)$, it can be seen that $\mathrm{Hom}_{\mathcal{T}}(N,M)=0$.
Suppose that $n:=\ell_{\mathcal{Y}}(N)>1$. Then, there exists a distinguished triangle

$$\eta_n:\qquad N_{n-1}\longrightarrow N\longrightarrow Y_n\longrightarrow N_{n-1}[1]$$

such that $N_{n-1}\in\mathfrak{F}(\mathcal{Y})$, $Y_n\in\mathcal{Y}$ and $\ell_{\mathcal{Y}}(N_{n-1})=n-1$. Applying $\mathrm{Hom}_{\mathcal{T}}(-,M)$ to the triangle η_n, we get the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(Y_n,M)\longrightarrow\mathrm{Hom}_{\mathcal{T}}(N,M)\longrightarrow\mathrm{Hom}_{\mathcal{T}}(N_{n-1},M).$$

By induction, we have that $\mathrm{Hom}_{\mathcal{T}}(N_{n-1},M)=0=\mathrm{Hom}_{\mathcal{T}}(Y_n,M)$, and therefore $\mathrm{Hom}_{\mathcal{T}}(N,M)=0$. $\square$

Corollary 4.4. *Let $\mathcal{X}$ be a class of objects in a triangulated category $\mathcal{T}$. Then ${}^{\perp}\mathcal{X}={}^{\perp}\mathfrak{F}(\mathcal{X})$.*

Proof. It is enough to prove that ${}^{\perp}\mathcal{X}\subseteq{}^{\perp}\mathfrak{F}(\mathcal{X})$, since the other inclusion ${}^{\perp}\mathfrak{F}(\mathcal{X})\subseteq{}^{\perp}\mathcal{X}$ follows easily from the fact that $\mathcal{X}\subseteq\mathfrak{F}(\mathcal{X})$.

Let $Y\in{}^{\perp}\mathcal{X}$ and $Z\in\mathfrak{F}(\mathcal{X})$. Then, by 4.3, it follows that $\mathrm{Hom}_{\mathcal{T}}(Y,Z)=0$, since $\mathrm{Hom}_{\mathcal{T}}(Y,-)|_{\mathcal{X}}=0$. Thus $Y\in{}^{\perp}\mathfrak{F}(\mathcal{X})$ proving that ${}^{\perp}\mathcal{X}={}^{\perp}\mathfrak{F}(\mathcal{X})$. $\square$

Lemma 4.5. *Let $\mathcal{T}$ be a triangulated category. If there are two distinguished triangles $Z\to Y\to\theta_1\to Z[1]$ and $Y\to X\to\theta_2\to Y[1]$ such that $\mathrm{Hom}_{\mathcal{T}}(\theta_2,\theta_1[1])=0$, then there exist two distinguished triangles as follows $Z\to W\to\theta_2\to Z[1]$ and $W\to X\to\theta_1\to W[1]$.*

Proof. Let $Z\to Y\to\theta_1\to Z[1]$ and $Y\to X\to\theta_2\to Y[1]$ be distinguished triangles such that $\mathrm{Hom}_{\mathcal{T}}(\theta_2,\theta_1[1])=0$. By co-base change (see 2.1), we have the following commutative diagram

$$\begin{tikzcd}
 & Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & \\
\theta_2[-1] \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \theta_2 \arrow[d, equal] \\
\theta_2[-1] \arrow[r] & \theta_1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \theta_2 \\
 & Z[1] \arrow[r, equal] & Z[1], &
\end{tikzcd}$$

where the rows and columns are distinguished triangles. Using the fact that $\mathrm{Hom}_{\mathcal{T}}(\theta_2,\theta_1[1])=0$, it follows that $\eta:\ \theta_1\to C\to\theta_2\to\theta_1[1]$ splits, and hence we get the following distinguished triangle $\eta':\ \theta_2\to C\to\theta_1\to\theta_2[1]$.

Then, by base change (see 2.1), we obtain the following commutative diagram

$$\begin{tikzcd}
 & \theta_1[-1] \arrow[r, equal] \arrow[d] & \theta_1[-1] \arrow[d] & \\
Z \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & \theta_2 \arrow[r] \arrow[d] & Z[1] \arrow[d, equal] \\
Z \arrow[r] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & Z[1] \\
 & \theta_1 \arrow[r, equal] & \theta_1, &
\end{tikzcd}$$

where the rows and columns are distinguished triangles. Hence, the required distinguished triangles are $Z \to W \to \theta_2 \to Z[1]$ and $W \to X \to \theta_1 \to W[1]$. $\square$

Lemma 4.6. *Let $\mathcal{T}$ be a triangulated category and $\theta \in \mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(\theta, \theta[1]) = 0$, and let $\eta = \{\eta_i : \ M_{i-1} \to M_i \to \theta \to M_{i-1}[1]\}_{i=1}^n$ be a family of distinguished triangles. Then, for each $k \in [1, n]$, there exists a distinguished triangle $\xi_k : \ M_0 \to M_k \to \theta^k \to M_0[1]$.*

Proof. We will proceed by induction on k. For $k = 1$, we have that $\xi_1 := \eta_1$ is the required triangle.

Let $k > 1$. Suppose we have ξ_{k-1}. By co-base change (see 2.1), we get the following commutative diagram

$$\begin{tikzcd}
 & M_0 \arrow[r, equal] \arrow[d] & M_0 \arrow[d] & \\
\theta[-1] \arrow[r] \arrow[d, equal] & M_{k-1} \arrow[r] \arrow[d] & M_k \arrow[r] \arrow[d] & \theta \arrow[d, equal] \\
\theta[-1] \arrow[r] & \theta^{k-1} \arrow[r] \arrow[d] & L_k \arrow[r] \arrow[d] & \theta \\
 & M_0[0] \arrow[r, equal] & M_0[1], &
\end{tikzcd}$$

where the rows and columns are distinguished triangles. Since $\mathrm{Hom}_{\mathcal{T}}(\theta, \theta[1]) = 0$, the lower triangle of the last diagram splits and hence $L_k \simeq \theta^k$. Therefore, the second column of the above diagram, is the required triangle ξ_k. $\square$

Let $\mathcal{T}$ be a triangulated category and $\Theta = \{\Theta(i)\}_{i=1}^t$ be a family of objects in $\mathcal{T}$. For a given Θ-filtration $\xi = \{\xi_k \ : \ M_{k-1} \to M_k \to X_k \to M_{k-1}[1]\}_{k=0}^n$

of $M \in \mathfrak{F}(\Theta)$, we shall denote by $[M:\Theta(i)]_\xi$ the **ξ-filtration multiplicity** of $\Theta(i)$ in M. That is $[M:\Theta(i)]_\xi$ is the cardinal of the set $\{k \in [0,n] \mid X_k \simeq \Theta(i)\}$. In general, the filtration multiplicity could be depending on a given Θ-filtration. Observe that $\ell_{\Theta,\xi}(M) = \sum_{k=1}^{t} [M:\Theta(i)]_\xi$.

Proposition 4.7. *Let $\Theta = \{\Theta(i)\}_{i=1}^t$ be a family of objects in a triangulated category $\mathcal{T}$, and let $\leq$ be a linear order on $[1,t]$ such that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ for all $j \geq i$. If ξ is a Θ-filtration of $M \in \mathfrak{F}(\Theta)$, then there is a Θ-filtration η of M and a family Ξ of distinguished triangles satisfying the following conditions.*

(a) $m(i) := [M:\Theta(i)]_\xi = [M:\Theta(i)]_\eta$ *for all* $i \in [1,t]$.
(b) *The family η is ordered, that is,*

$$\eta = \{\eta_i \ : \ M_{i-1} \to M_i \to \Theta(k_i) \to M_{i-1}[1]\}_{i=0}^n$$

with $\Theta(k_0) := 0$, $M_{-1} := 0$ and $k_n \leq k_{n-1} \leq \cdots \leq k_1$ in $([1,t], \leq)$.
(c) $\Xi = \{\Xi_i \ : \ M'_{i-1} \to M'_i \to \Theta(\lambda_i)^{m(\lambda_i)} \to M'_{i-1}[1]\}_{i=0}^d$, $\{\Theta(\lambda_i)\}_{i=1}^d$ *is the set consisting of the different $\Theta(j)$ appearing in the Θ-filtration ξ of M. Moreover $\Theta(\lambda_0) := 0$, $M'_{-1} := 0$, $M'_d = M$ and $\lambda_d < \lambda_{d-1} < \cdots < \lambda_1$ in $([1,t], \leq)$,*

Proof. Let ξ be a Θ-filtration of $M \in \mathfrak{F}(\Theta)$. We can assume that $M \neq 0$ since the result is trivial in this case.

We start by proving (a) and (b), proceeding by induction on $n := \ell_{\Theta,\xi}(M)$. If $n = 1$, the Θ-filtration ξ is already ordered and hence $\eta := \xi$ satisfies the required properties. Let $n \geq 2$ and $\xi := \{\xi_i \ : \ M_{i-1} \to M_i \to \Theta(k_i) \to M_{i-1}[1]\}_{i=0}^n$ be the Θ-filtration of M. Since $\xi' := \xi - \{\xi_n\}$ is a Θ-filtration of M_{n-1} and $\ell_{\Theta,\xi'}(M_{n-1}) = n-1$, by induction there is an ordered Θ-filtration $\eta' = \{\eta'_i \ : \ M'_{i-1} \to M'_i \to \Theta(k'_i) \to M'_{i-1}[1]\}_{i=0}^{n-1}$ of M_{n-1} with $k'_{n-1} \leq k'_{n-2} \leq \cdots \leq k'_1$ and $[M_{n-1}:\Theta(i)]_{\xi'} = [M_{n-1}:\Theta(i)]_{\eta'}$ $\forall\, i$. If $k_n \leq k'_{n-1}$ then $\eta := \eta' \cup \{\xi_n\}$ satisfies the required conditions.
Suppose now that $k'_{n-1} < k_n$. Let $l := \max\{m \in [1, n-1] \mid k'_{n-m} < k_n\}$. Observe that the Θ-filtration $\eta' \cup \{\xi_n\}$ is almost the one we want, the only triangle that does not have its ordered multiplicity is precisely the ξ_n. This can be rearranged by applying l-times 4.5 to $\eta' \cup \{\xi_n\}$.

(c) In order to construct Ξ, we use the ordered Θ-filtration η from (b). We proceed as follows. For each $i \in [1,n]$, we group the k_i that are the same and rename them by λ_i. So we get $\lambda_d < \lambda_{d-1} < \cdots < \lambda_1$ on $([1,t], \leq)$, and hence $\Theta(\lambda_1), \cdots, \Theta(\lambda_d)$ are the different $\Theta(j)$ appearing in the Θ-filtration η of M. Define $s(i) := m(\lambda_i) = [M:\Theta(\lambda_i)]$, $\alpha(i) := \sum_{j=1}^{i} s(i)$ and $\alpha(0) := -1$.
We divide the filtration η into the following pieces

$$\{\eta_i \ : \ M_{i-1} \longrightarrow M_i \longrightarrow \theta(\lambda_l) \longrightarrow M_{i-1}[1]\ \}_{i=\alpha(l-1)+1}^{\alpha(l)},$$

with $l \in [1, d]$. By 4.6, for each $l \in [1, d]$, we obtain the following distinguished triangle

$$\Xi_l : \quad M_{\alpha(l-1)} \longrightarrow M_{\alpha(l)} \longrightarrow \theta(\lambda_l)^{s(l)} \longrightarrow M_{\alpha(l-1)[1]}.$$

Hence, by setting $\Xi_0 := \eta_0$ and $M'_i := M_{\alpha(i-1)}$ for $i \in [1, d]$, we conclude that the filtration $\Xi = \{\Xi_i\}_{i=0}^{d}$ satisfies the required properties. $\Box$

Let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be a family of objects in a triangulated category $\mathcal{T}$. We denote by $\Theta^{\oplus}$ the subcategory of $\mathcal{T}$, whose objects are the finite direct sums of copies of objects in Θ.

Lemma 4.8. *Let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be a family of objects in a triangulated category $\mathcal{T}$. Then, the following statements hold.*

(a) $\mathfrak{F}(\Theta) = \mathfrak{F}(\Theta^{\oplus})$.

(b) *If $\mathcal{T}$ is an artin triangulated R-category, then $\Theta^{\oplus}$ is functorially finite.*

Proof. (a) Since $\Theta \subseteq \Theta^{\oplus}$, it follows that $\mathfrak{F}(\Theta) \subseteq \mathfrak{F}(\Theta^{\oplus})$.

Let $M \in \mathfrak{F}(\Theta^{\oplus})$. We prove, by induction on $m := \ell_{\Theta^{\oplus}}(M)$, that $M \in \mathfrak{F}(\Theta)$. If $m = 1$, then $M \in \Theta^{\oplus}$ and hence $M = \oplus_{i=1}^{n} \Theta(k_i)^{m_i}$. Since $\mathfrak{F}(\Theta)$ is closed under extensions (see 4.2) and $\Theta(k_i) \in \mathfrak{F}(\Theta)$, we get that $M \in \mathfrak{F}(\Theta)$.
Let $m > 1$. Then, there is a distinguished triangle $M_{m-1} \to M \to \Theta(k_m)^{\lambda(m)} \to M_{m-1}[1]$ with $\ell_{\Theta^{\oplus}}(M_{m-1}) = m-1$. Hence, by induction, we get that $M_{m-1} \in \mathfrak{F}(\Theta)$. Therefore $M \in \mathfrak{F}(\Theta)$ since $\mathfrak{F}(\Theta)$ is closed under extensions.

(b) The proof given in [4, Proposition 4.2] can be easily extended to the context of an artin triangulated R-category. $\Box$

Lemma 4.9. *Let $\mathcal{X}$ be a class of objects in a triangulated category $\mathcal{T}$ such that $0 \in \mathcal{X}$ and $\mathcal{X}$ is closed under isomorphisms. Then $\mathfrak{F}_n(\mathcal{X}) = *_{i=1}^{n} \mathcal{X}$ for $n \geq 1$, and $\mathfrak{F}_k(\mathcal{X}) \subseteq \mathfrak{F}_{k+1}(\mathcal{X})$ for any $k \in \mathbb{N}$.*

Proof. We have that $\mathfrak{F}_0(\mathcal{X}) := \{0\}$ and $\mathcal{X} \subseteq \mathfrak{F}_1(\mathcal{X}) = \{0\} * \mathcal{X}$. On the other hand, since $\mathcal{X}$ is closed under isomorphism, then $\{0\} * \mathcal{X} \subseteq \mathcal{X}$. Hence, $\mathfrak{F}_1(\mathcal{X}) = \mathcal{X}$ and so $\mathfrak{F}_2(\mathcal{X}) = \mathcal{X} * \mathcal{X}$. Continuing in the same way, we get that $\mathfrak{F}_n(\mathcal{X}) = *_{i=1}^{n} \mathcal{X}$ for $n \geq 1$. Therefore, using the fact that the operation $*$ is associative, it follows that $\mathfrak{F}_{k+1}(\mathcal{X}) = \mathcal{X} * \mathfrak{F}_k(\mathcal{X})$. Since $0 \in \mathcal{X}$, we conclude that $\mathfrak{F}_k(\mathcal{X}) \subseteq \mathfrak{F}_{k+1}(\mathcal{X})$ for any $k \in \mathbb{N}$. $\Box$

The following result is a generalization, for triangulated categories, of the Ringel's result [30, Theorem 1]. The proof we give here uses the triangulated version of Gentle-Todorov's theorem due to Xiao-Wu Chen [12, Theorem 1.3].

Theorem 4.10. *Let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be a family of objects in an artin triangulated R-category $\mathcal{T}$, and let $\leq$ be a linear order on the set $[1, t]$ such that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ for all $j \geq i$. Then $\mathfrak{F}(\Theta) = *_{i=1}^{t} \Theta^{\oplus}$ and it is functorially finite.*

Proof. Let $\mathcal{X} := \Theta^{\oplus}$. We assert that $\mathfrak{F}_t(\mathcal{X}) = \mathfrak{F}(\Theta)$. Indeed, by 4.8, we have that $\mathfrak{F}(\mathcal{X}) = \mathfrak{F}(\Theta)$ and hence $\mathfrak{F}(\Theta) = \cup_{n\in\mathbb{N}}\mathfrak{F}_n(\mathcal{X})$.
Let $M \in \mathfrak{F}(\Theta)$, and consider a Θ-filtration ξ of M. Then, by 4.7(c), there is a family of distinguished triangles

$$\Xi = \{\Xi_i :\ M'_{i-1} \longrightarrow M'_i \longrightarrow \Theta(\lambda_i)^{m(\lambda_i)} \longrightarrow M'_{i-1}[1]\ \}_{i=0}^{d},$$

where $\{\Theta(\lambda_i)\}_{i=1}^{d}$ is the set of the different $\Theta(j)$ appearing in the Θ-filtration ξ of M, $\lambda_d < \lambda_{d-1} < \cdots < \lambda_1$ and $M'_d = M$. Therefore $M \in \mathfrak{F}_d(\mathcal{X})$ with $d \leq t$. Since $\mathcal{X}$ is closed under isomorphisms and contain the zero object, by 4.9, it follows that $\mathfrak{F}_d(\mathcal{X}) \subseteq \mathfrak{F}_t(\mathcal{X})$. Thus $\mathfrak{F}(\Theta) \subseteq \mathfrak{F}_t(\mathcal{X})$ and hence $\mathfrak{F}_t(\mathcal{X}) = \mathfrak{F}(\Theta)$, proving our assertion.
By 4.8 (b), we know that $\mathcal{X}$ is functorially finite. Furthermore, from 4.9 and the assertion above, it follows that $\mathfrak{F}(\Theta) = *_{i=1}^{t}\mathcal{X}$. Hence the result follows from [12, Theorem 1.3] and its dual. $\square$

Definition 4.11. *Let $\Theta = \{\Theta(i)\}_{i=1}^{n}$ be a family of objects in a triangulated category $\mathcal{T}$. The* Θ**-projective** *objects in $\mathcal{T}$ is the class $\mathcal{P}(\Theta) := {}^{\perp}\mathfrak{F}(\Theta)[1]$. Dually, the* Θ**-injective** *objects in $\mathcal{T}$ is the class $\mathcal{I}(\Theta) := \mathfrak{F}(\Theta)^{\perp}[-1]$.*

Observe that, by 4.4 and its dual, we have that $\mathcal{P}(\Theta) = {}^{\perp}\Theta[1]$ and $\mathcal{I}(\Theta) = \Theta^{\perp}[-1]$.

In what follows, we use Ringel's ideas, in the paper [30], to proof that under certain conditions, $\mathcal{P}(\Theta)$ is a precovering class and $\mathcal{I}(\Theta)$ is a preenveloping one. To do that, we use the following two lemmas (compare with [30, Lemma 3 and Lemma 4]).

Lemma 4.12. *Let $\Theta = \{\Theta(i)\}_{i=1}^{n}$ be a family of objects, in a Hom-finite triangulated R-category $\mathcal{T}$, such that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i)[1]) = 0$ for $j \geq i$. Consider $t \in [1,n]$ and $N \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),N[1]) = 0$ for $j > t$. Then, there exists a distinguished triangle in $\mathcal{T}$*

$$N \longrightarrow N_t \longrightarrow \Theta(t)^m \longrightarrow N[1]\ ,$$

where $m := \ell_R\mathrm{Hom}_{\mathcal{T}}(\Theta(t),N[1])$ and $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),N_t[1]) = 0$ for $j \geq t$.

Proof. If $\mathrm{Hom}_{\mathcal{T}}(\Theta(t),N[1]) = 0$, the distinguished triangle we are looking for is $N \xrightarrow{1} N \to 0 \to N[1]$.

Let $\mathrm{Hom}_{\mathcal{T}}(\Theta(t),N[1]) \neq 0$. Then, by the dual of 3.4, there is a distinguished triangle

$$\eta :\ N \longrightarrow N_t \longrightarrow \Theta(t)^m \xrightarrow{\ h\ } N[1]$$

such that the map $\mathrm{Hom}_{\mathcal{T}}(\Theta(t),h) : \mathrm{Hom}_{\mathcal{T}}(\Theta(t),\Theta(t)^m) \to \mathrm{Hom}_{\mathcal{T}}(\Theta(t),N[1])$ is surjective. Applying $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),-)$ to η, we get the following exact sequence

$$(\Theta(j),\Theta(t)^m) \longrightarrow (\Theta(j),N[1]) \longrightarrow (\Theta(j),N_t[1]) \longrightarrow (\Theta(j),\Theta(t)^m[1]).$$

Since $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), \Theta(t)[1]) = 0$ for $j \geq t$ and $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N[1]) = 0$ for $j > t$, it follows that $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N_t[1]) = 0$ for $j > t$. For $j = t$, we know that $\operatorname{Hom}_{\mathcal{T}}(\Theta(t), h)$ is an epimorphism and hence $\operatorname{Hom}_{\mathcal{T}}(\Theta(t), N_t[1]) = 0$; proving the lemma. $\Box$

Lemma 4.13. *Let $\Theta = \{\Theta(i)\}_{i=1}^n$ be a family of objects, in a Hom-finite triangulated R-category $\mathcal{T}$, such that $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ for $j \geq i$. Consider $t \in [1, n]$ and $N \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N[1]) = 0$ for $j > t$. Then, there exists a distinguished triangle in $\mathcal{T}$*

$$N \longrightarrow Y \longrightarrow X \longrightarrow N[1]$$

with $X \in \mathfrak{F}(\{\Theta(i) \mid i \in [1, t]\})$ and $Y \in \mathcal{I}(\Theta)$.

Proof. Since $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N[1]) = 0$ for $j > t$, it follows from 4.12 the existence of a distinguished triangle

$$\eta_{t+1} : \ N \xrightarrow{\mu_t} N_t \longrightarrow Q_t \longrightarrow N[1]$$

with $Q_t := \Theta(t)^{m_t}$ and $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N_t[1]) = 0$ for $j \geq t$. Similarly, there is a distinguished triangle

$$\eta_t : \ N_t \xrightarrow{\mu_{t-1}} N_{t-1} \longrightarrow Q_{t-1} \longrightarrow N_t[1]$$

with $Q_{t-1} := \Theta(t-1)^{m_{t-1}}$ and $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N_{t-1}[1]) = 0$ for $j \geq t-1$. Continuing this procedure, we get distinguished triangles

$$\eta_i : \ N_i \xrightarrow{\mu_{i-1}} N_{i-1} \longrightarrow Q_{i-1} \longrightarrow N_i[1]$$

with $Q_{i-1} := \Theta(i-1)^{m_{i-1}}$ and $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N_i[1]) = 0$ for $j \geq i$. In what follows, for $\alpha_r := \mu_{t-r} \ldots \mu_{t-1}\mu_t$ with $0 \leq r \leq t-1$, we will construct, inductively, distinguished triangles

$$\xi_r : \ N \xrightarrow{\alpha_r} N_{t-r} \longrightarrow X_{t-r} \longrightarrow N[1]$$

with $X_{t-r} \in \mathfrak{F}(\{\Theta(i) \mid i \in [t-r, t]\})$ and $\operatorname{Hom}_{\mathcal{T}}(\Theta(j), N_{t-r}[1]) = 0$ for $j \geq t-r$. If $r = 0$, we set $\xi_0 := \eta_{t+1}$. Suppose that $r > 0$ and that the triangle ξ_r is already constructed. Consider the following diagram of co-base

change (see 2.1)

$$\begin{tikzcd}
 & \Theta(t-r-1)^{m_{t-r-1}}[-1] \arrow[r, equal] \arrow[d] & \Theta(t-r-1)^{m_{t-r-1}}[-1] \arrow[d] & \\
N \arrow[r, "\alpha_r"] \arrow[d, equal] & N_{t-r} \arrow[r] \arrow[d, "\mu_{t-r-1}"] & X_{t-r} \arrow[r] \arrow[d] & N[1] \arrow[d, equal] \\
N \arrow[r, "\alpha_{r+1}"] & N_{t-r-1} \arrow[r] \arrow[d] & X_{t-r-1} \arrow[r] \arrow[d] & N[1] \\
 & \Theta(t-r-1)^{m_{t-r-1}} \arrow[r, equal] & \Theta(t-r-1)^{m_{t-r-1}}. &
\end{tikzcd}$$

By induction, we have that $X_{t-r} \in \mathfrak{F}(\{\Theta(i) \mid i \in [t-r,t]\})$. Thus, $\Theta(t-r-1)^{m_{t-r-1}}$, $X_{t-r} \in \mathfrak{F}(\{\Theta(i) \mid i \in [t-r-1,t]\})$. Since $\mathfrak{F}(\{\Theta(i) \mid i \in [t-r-1,t]\})$ is closed under extensions, it follows that $X_{t-r-1} \in \mathfrak{F}(\{\Theta(i) \mid i \in [t-r-1,t]\})$. Moreover $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), N_{t-r-1}[1]) = 0$ for $j \geq t-r-1$. Therefore ξ_{r+1} is the triangle from the second row of the last diagram. Then, the required triangle is ξ_{t-1}. $\square$

Theorem 4.14. *Let $\Theta = \{\Theta(i)\}_{i=1}^n$ be a family of objects in an artin triangulated R-category $\mathcal{T}$, and let $\leq$ be a linear order on the set $[1,t]$ such that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ for all $j \geq i$. Then, the following statements holds.*

(a) *For any object $X \in \mathcal{T}$ there are two distinguished triangles in $\mathcal{T}$*

$$X \longrightarrow Y_X \longrightarrow C_X \longrightarrow X[1] \qquad \text{with } Y_X \in \mathcal{I}(\Theta),\ C_X \in \mathfrak{F}(\Theta),$$

$$X[-1] \longrightarrow K_X \longrightarrow Q_X \longrightarrow X \qquad \text{with } Q_X \in \mathcal{P}(\Theta),\ K_X \in \mathfrak{F}(\Theta).$$

(b) *$\mathcal{P}(\Theta)$ is a precovering class and $\mathcal{I}(\Theta)$ is a preenveloping one in $\mathcal{T}$.*

Proof. (a) For simplicity, we assume that the linear order $\leq$ on the set $[1,t]$ is the natural one. Furthermore, we only prove the existence of the first triangle, since the existence of the other one follows by duality. Let $X \in \mathcal{T}$ and $t := n$. Then, from 4.13, we get a distinguished triangle $X \to Y_X \to C_X \to X[1]$ in $\mathcal{T}$ such that $Y_X \in \mathcal{I}(\Theta)$ and $C_X \in \mathfrak{F}(\Theta)$.

(b) We start proving that $\mathcal{I}(\Theta)$ is a preenveloping class in $\mathcal{T}$. Indeed, let $X \in \mathcal{T}$. Then, by (a), there is a distinguished triangle

$$X \xrightarrow{\beta} Y_X \longrightarrow C_X \longrightarrow X[1]$$

with $Y_X \in \mathcal{I}(\Theta)$ and $C_X \in \mathfrak{F}(\Theta)$. We claim that β is an $\mathcal{I}(\Theta)$-preenvelope of X. To see that, we consider a morphism $\beta' : X \to Y'$ with $Y' \in \mathcal{I}(\Theta)$. Then,

by co-base change (see 2.1), we have the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
C_X[-1] \arrow[r] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "\beta'"] & Y_X \arrow[r] \arrow[d, "\gamma"] & C_X \arrow[d, equal] \\
C_X[-1] \arrow[r, "\alpha"] & Y' \arrow[r, "u"] & L \arrow[r] & C_X,
\end{tikzcd}$$

where the rows are distinguished triangles. Since $Y' \in \mathcal{I}(\Theta) = \mathfrak{F}(\Theta)^{\perp}[-1]$ and $C_X \in \mathfrak{F}(\Theta)$, we get that $\mathrm{Hom}_{\mathcal{T}}(C_X[-1], Y') = 0$. Therefore $\alpha = 0$ and thus β' factors through β; proving that β is an $\mathcal{I}(\Theta)$-preenvelope of X. Finally, the proof that $\mathcal{P}(\Theta)$ is a precovering class in $\mathcal{T}$ is rather similarly by using the second triangle in (a). $\square$

5. Homological systems

In this section, we introduce several homological systems of objects in a triangulated category $\mathcal{T}$, over a linearly ordered finite set. This homological systems generalize the notion of stratifying systems (see [16, 23, 24, 25]) in a module category. We recall that $[1, n] := \{1, 2, \cdots, n\}$ for any $n \in \mathbb{Z}^+$.

Definition 5.1. *A* **Θ-system** $(\Theta, \leq)$ *of size t, in a triangulated category $\mathcal{T}$, consists of the following data.*

- (S1) $\leq$ *is a linear order on* $[1, t]$.
- (S2) $\Theta = \{\Theta(i)\}_{i=1}^t$ *is a family of indecomposable objects in $\mathcal{T}$.*
- (S3) $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)) = 0$ *for $j > i$.*
- (S4) $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ *for $j \geq i$.*
- (S5) $\mathrm{Hom}_{\mathcal{T}}(\Theta, \Theta[-1]) = 0$.

Definition 5.2. *A* **Θ-projective system** $(\Theta, \mathbf{Q}, \leq)$ *of size t, in a triangulated category $\mathcal{T}$, consists of the following data.*

- (PS1) $\leq$ *is a linear order on* $[1, t]$.
- (PS2) $\Theta = \{\Theta(i)\}_{i=1}^t$ *is a family of non-zero objects in $\mathcal{T}$.*
- (PS3) $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)) = 0$ *for $j > i$.*
- (PS4) $\mathbf{Q} = \{Q(i)\}_{i=1}^t$ *is a family of indecomposable objects in $\mathcal{T}$ such that* $Q := \bigoplus_{i=1}^t Q(i) \in {}^{\perp}\Theta[-1] \cap {}^{\perp}\Theta[1]$.
- (PS5) *For every $i \in [1, t]$, there exists a distinguished triangle in $\mathcal{T}$*

$$\eta_i \ : \ K(i) \longrightarrow Q(i) \xrightarrow{\beta_i} \Theta(i) \longrightarrow K(i)[1]$$

such that $K(i) \in \mathfrak{F}(\{\Theta(j) \mid j > i\})$ and $\mathrm{Hom}_{\mathcal{T}}(K(i)[1], \Theta(i)) = 0$.

Definition 5.3. *A* **Θ-injective system** $(\Theta, \mathbf{Y}, \leq)$ *of size t, in a triangulated category $\mathcal{T}$, consists of the following data.*

- (IS1) $\leq$ *is a linear order on* $[1, t]$.
- (IS2) $\Theta = \{\Theta(i)\}_{i=1}^t$ *is a family of non-zero objects in $\mathcal{T}$.*

(IS3) $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i))=0$ *for* $j>i$.

(IS4) $\mathbf{Y}=\{Y(i)\}_{i=1}^{t}$ *is a family of indecomposable objects in* $\mathcal{T}$ *such that* $Y:=\bigoplus_{i=1}^{t}Y(i)\in\Theta^{\perp}[-1]\cap\Theta^{\perp}[1]$.

(IS5) *For every* $i\in[1,t]$*, there exists a distinguished triangle in* $\mathcal{T}$

$$\xi_i\ :\ Z(i)[-1]\longrightarrow\Theta(i)\xrightarrow{\alpha_i}Y(i)\longrightarrow Z(i)$$

such that $Z(i)\in\mathfrak{F}(\{\Theta(j)\mid j<i\})$ *and* $\mathrm{Hom}_{\mathcal{T}}(\Theta(i),Z(i)[-1])=0$.

Remark 5.4. *A triple* $(\Theta,\mathbf{Y},\leq)$ *is a* Θ*-injective system of size* t*, in a triangulated category* $\mathcal{T}$*, if and only if* $(\Theta^{op},\mathbf{Y}^{op},\leq^{op})$ *is a* Θ^{op}*-projective system of size* t *in the opposite triangulated category* $\mathcal{T}^{op}$*, where* $\leq^{op}$ *is the opposite order of* $\leq$ *in* $[1,t]$*. Therefore, any obtained result for* Θ*-projective systems can be transfered to the* Θ*-injective systems, and so, we could be dealing only with* Θ*-projective systems.*

Proposition 5.5. *Let* $(\Theta,\mathbf{Q},\leq)$ *be a* Θ*-projective system of size* t*, in a triangulated category* $\mathcal{T}$*. Then, the following conditions hold.*

(a) $\mathrm{Hom}_{\mathcal{T}}(K(j),\Theta(i))=0=\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i)[1])$ *for all* $j\geq i$.

(b) $\mathrm{Hom}_{\mathcal{T}}(\beta_j,\Theta(i)):\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i))\to\mathrm{Hom}_{\mathcal{T}}(Q(j),\Theta(i))$ *is an isomorphism of abelian groups, for all* $j\geq i$.

(c) *If* $\mathrm{Hom}_{\mathcal{T}}(K(j)[2],\Theta(i))=0\ \ \forall\, i,j\in[1,t]$*, then* $\mathrm{Hom}_{\mathcal{T}}(\Theta,\Theta[-1])=0$.

Proof. (a) Let $j\geq i$. Using the fact that $K(j)\in\mathfrak{F}(\{\Theta(\lambda)\mid\lambda>j\})$ and since $\mathrm{Hom}_{\mathcal{T}}(\Theta(\lambda),\Theta(i))=0$ for $\lambda>j\geq i$, it follows from 4.3 that $\mathrm{Hom}_{\mathcal{T}}(K(j),\Theta(i))=0$. Consider the distinguished triangle given in 5.2 (PS5)

$$\eta_j\ :\ K(j)\longrightarrow Q(j)\xrightarrow{\beta_j}\Theta(j)\longrightarrow K(j)[1]\ .$$

Applying $\mathrm{Hom}_{\mathcal{T}}(-,\Theta(i)[1])$ to η_j, we get the exact sequence

$$(K(j)[1],\Theta(i)[1])\longrightarrow(\Theta(j),\Theta(i)[1])\longrightarrow(Q(j),\Theta(i)[1])\ .$$

Thus, since $\mathbf{Q}\subseteq{}^{\perp}\Theta[1]$ and $\mathrm{Hom}_{\mathcal{T}}(K(j),\Theta(i))=0$, it follows from the sequence above that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i)[1])=0$ for $j\geq i$.

(b) Let $j\geq i$. Applying $\mathrm{Hom}_{\mathcal{T}}(-,\Theta(i))$ to the above distinguished triangle η_j, we get the exact sequence

$$(K(j)[1],\Theta(i))\longrightarrow(\Theta(j),\Theta(i))\xrightarrow{(\beta_j,\Theta(i))}(Q(j),\Theta(i))\longrightarrow(K(j),\Theta(i)).$$

We have that $\mathrm{Hom}_{\mathcal{T}}(\beta_j,\Theta(i))$ is an epimorphism, since by (a) we know that $\mathrm{Hom}_{\mathcal{T}}(K(j),\Theta(i))=0$ for $j\geq i$. Since $\mathrm{Hom}_{\mathcal{T}}(K(i)[1],\Theta(i))=0$ (see 5.2 (PS5)), we conclude that $\mathrm{Hom}_{\mathcal{T}}(\beta_i,\Theta(i))$ is an isomorphism.
Assume that $j>i$. Then $\mathrm{Ker}\,(\mathrm{Hom}_{\mathcal{T}}(\beta_j,\Theta(i)))\subseteq\mathrm{Hom}_{\mathcal{T}}(\Theta(j),\Theta(i))=0$ and hence $\mathrm{Hom}_{\mathcal{T}}(\beta_j,\Theta(i))$ is also an isomorphism.

(c) Let $i, j \in [1, t]$. Applying $\mathrm{Hom}_{\mathcal{T}}(-, \Theta(i)[-1])$ to the above distinguished triangle η_j, we get the exact sequence

$$(K(j)[1], \Theta(i)[-1]) \longrightarrow (\Theta(j), \Theta(i)[-1]) \longrightarrow (Q(j), \Theta(i)[-1])\ .$$

Using the fact that $\mathbf{Q} \subseteq {}^{\perp}\Theta[-1]$ and since $\mathrm{Hom}_{\mathcal{T}}(K(j)[2], \Theta(i)) = 0$, it follows that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[-1]) = 0$; proving that $\mathrm{Hom}_{\mathcal{T}}(\Theta, \Theta[-1]) = 0$. □

Proposition 5.6. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in an artin triangulated R-category $\mathcal{T}$. Then, the following statements hold.*

(a) *For each $i \in [1, t]$, the morphism $\beta_i : Q(i) \to \Theta(i)$, appearing in the triangle η_i from 5.2 (PS5), is a $\mathcal{P}(\Theta)$-cover of $\Theta(i)$.*

(b) *Let $(\Theta, \mathbf{Q'}, \leq)$ be another Θ-projective system of size t, in $\mathcal{T}$. Then $\mathbf{Q'} \simeq \mathbf{Q}$; that is, for each $i \in [1, t]$, there is an isomorphism $\rho_i : Q(i) \to Q'(i)$ such that the following diagram in $\mathcal{T}$ commutes*

$$\begin{tikzcd} Q(i) \arrow[rr, "\rho_i"] \arrow[dr, "\beta_i"'] & & Q'(i) \arrow[dl, "\beta'_i"] \\ & \Theta(i). & \end{tikzcd}$$

Proof. (a) Let $i \in [1, t]$. We start by proving that $\beta_i : Q(i) \to \Theta(i)$ is right minimal. Firstly, we assert that $\beta_i \neq 0$. Indeed, by 5.5 (b), we have that $\mathrm{Hom}_{\mathcal{T}}(\beta_i, \Theta(i)) : \mathrm{End}_{\mathcal{T}}(\Theta(i)) \to \mathrm{Hom}_{\mathcal{T}}(Q(i), \Theta(i))$ is an isomorphism. Thus $\beta_i = \mathrm{Hom}_{\mathcal{T}}(\beta_i, \Theta(i))(1_{\Theta(i)}) \neq 0$ since $1_{\Theta(i)} \neq 0$. Let $f : Q(i) \to Q(i)$ be such that $\beta_i f = \beta_i$. Then $\beta_i = \beta_i f^n \quad \forall n \in \mathbb{N}^+$. Since $\beta_i \neq 0$, it follows that $f^n \neq 0\ \forall n \in \mathbb{N}^+$. Using the fact that $Q(i)$ is indecomposable, we get from 3.3 (a), that $\mathrm{End}_{\mathcal{T}}(Q(i))$ is a local artin R-algebra. Thus, $\mathrm{rad}\,(\mathrm{End}_{\mathcal{T}}(Q(i)))$ is nilpotent and coincides with the set of non-invertible elements of $\mathrm{End}_{\mathcal{T}}(Q(i))$. Since $f^n \neq 0\ \ \forall n \in \mathbb{N}^+$, we conclude that $f \notin \mathrm{rad}\,(\mathrm{End}_{\mathcal{T}}(Q(i)))$ and therefore f is invertible; proving that $\beta_i : Q(i) \to \Theta(i)$ is right minimal.
Finally, we prove that $\beta_i : Q(i) \to \Theta(i)$ is a $\mathcal{P}(\Theta)$-precover of $\Theta(i)$. Let $g : X \to \Theta(i)$ be in $\mathcal{T}$, with $X \in \mathcal{P}(\Theta)$. Applying $\mathrm{Hom}_{\mathcal{T}}(X, -)$ to the distinguished triangle η_i from 5.2 (PS5), we get the exact sequence

$$(X, K(i)) \longrightarrow (X, Q(i)) \xrightarrow{(X, \beta_i)} (X, \Theta(i)) \longrightarrow (X, K(i)[1]).$$

Since $X \in \mathcal{P}(\Theta)$ and $K(i) \in \mathfrak{F}(\Theta)$, we conclude that $\mathrm{Hom}_{\mathcal{T}}(X, K(i)[1]) = 0$; proving that g factorizes through β_i, and thus β_i is a $\mathcal{P}(\Theta)$-precover of $\Theta(i)$.

(b) It is immediate from (a) □

Let $(\Theta, \leq)$ be Θ-system in a triangulated category $\mathcal{T}$. A natural question here, is to ask for the existence of a family $\mathbf{Q}$ of objects in $\mathcal{T}$ such that $(\Theta, \mathbf{Q}, \leq)$ is a Θ-projective system. In order to do that, we will need the

following results. Recall that, for any $a, b \in \mathbb{Z}$ with $a \leq b$, we set $[a, b] := \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

Lemma 5.7. *Let $(\Theta, \leq)$ be a Θ-system of size t, in a triangulated category $\mathcal{T}$, where $\leq$ is the natural order on the set $[1, t]$. Then, the following statements hold.*

(a) *If $M \in \mathfrak{F}(\{\Theta(j) \mid j \in [i, i+k]\})$, $N \in \mathfrak{F}(\{\Theta(r) \mid r > i+k\})$ and $L \in \mathfrak{F}(\{\Theta(s) \mid s < i\})$, then $\mathrm{Hom}_{\mathcal{T}}(N, M) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(M, L) = 0$.*
(b) *If $M \in \mathfrak{F}(\{\Theta(j) \mid j \in [i, i+k]\})$, $N \in \mathfrak{F}(\{\Theta(r) \mid r \geq i+k\})$ and $L \in \mathfrak{F}(\{\Theta(s) \mid s \leq i\})$, then $\mathrm{Hom}_{\mathcal{T}}(N, M[1]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(M, L[1]) = 0$.*
(c) *If $M, N \in \mathfrak{F}(\Theta)$ then $\mathrm{Hom}_{\mathcal{T}}(M, N[-1]) = 0$.*

Proof. It follows immediately from 4.3 and the definition of stratifying system. □

Proposition 5.8. *Let $(\Theta, \leq)$ be a Θ-system of size t, in an artin triangulated R-category $\mathcal{T}$, and let $\leq$ be the natural order on $[1, t]$, $t > 1$ and $i \in [1, t]$. Then, for each $k \in [1, t-i]$, there exists a distinguished triangle in $\mathcal{T}$*

$$\xi_k \ : \ V_k \longrightarrow U_k \longrightarrow \Theta(i) \longrightarrow V_k[1]$$

satisfying the following conditions:

(a) *U_k is indecomposable,*
(b) *$V_k \in \mathfrak{F}(\{\Theta(j) \mid i < j \leq i+k\})$,*
(c) *$\mathrm{Hom}_{\mathcal{T}}(U_k, \Theta(j)[1]) = 0$ for $j \in [i, i+k]$.*

Proof. We will proceed by induction on k.
Let $k = 1$. By definition, we have that $\mathrm{Hom}_{\mathcal{T}}(\Theta(i+1), \Theta(i)) = 0$. If $\mathrm{Hom}_{\mathcal{T}}(\Theta(i), \Theta(i+1)[1]) = 0$, the desired triangle is the following

$$0 \longrightarrow \Theta(i) \xrightarrow{1} \Theta(i) \longrightarrow 0.$$

Suppose that $\mathrm{Hom}_{\mathcal{T}}(\Theta(i), \Theta(i+1)[1]) \neq 0$. Then, by 3.4, there exists a not splitting distinguished triangle in $\mathcal{T}$

$$\xi \ : \ \Theta(i+1)^n \longrightarrow E \longrightarrow \Theta(i) \longrightarrow \Theta(i+1)^n \ ;$$

and moreover, we have that $\mathrm{Hom}_{\mathcal{T}}(E, \Theta(i+1)[1]) = 0$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, \Theta(i)[1])$ to ξ, we get the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(\Theta(i), \Theta(i)[1]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, \Theta(i)[1]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(\Theta(i+1)^n, \Theta(i)[1]).$$

Since $\mathrm{Hom}_{\mathcal{T}}(\Theta(i), \Theta(i)[1]) = 0 = \mathrm{Hom}_{\mathcal{T}}(\Theta(i+1)^n, \Theta(i)[1]) = 0$, we conclude that $\mathrm{Hom}_{\mathcal{T}}(E, \Theta(i)[1]) = 0$. Moreover, since $\mathrm{Hom}_{\mathcal{T}}(\Theta(i+1)^n, \Theta(i)) = 0 = \mathrm{Hom}_{\mathcal{T}}(\Theta(i+1)^n, \Theta(i)[-1]) = 0$, it follows by 3.5 the existence of a distinguished triangle

$$\xi' : \ \Theta(i+1)^m \longrightarrow U_1 \longrightarrow \Theta(i) \longrightarrow \Theta(i+1)^m[1]$$

with $m \leq n$ and U_1 an indecomposable direct summand of E. Thus, the distinguished triangle $\xi_1 := \xi'$ satisfies the required conditions. Suppose now that there exists a distinguished triangle

$$\xi_k : \quad V_k \longrightarrow U_k \longrightarrow \Theta(i) \longrightarrow V_k[1]$$

satisfying the above required properties. We construct the distinguished triangle ξ_{k+1}, from ξ_k, as follows. If $\mathrm{Hom}_{\mathcal{T}}(U_k, \Theta(i+k+1)[1]) = 0$, the triangle $\xi_{k+1} := \xi_k$ is the desired one.

Suppose that $\mathrm{Hom}_{\mathcal{T}}(U_k, \Theta(i+k+1)[1]) \neq 0$. Then, by 3.4, there exists a not splitting distinguished triangle in $\mathcal{T}$

$$\eta : \quad \Theta(i+k+1)^a \longrightarrow U \longrightarrow U_k \longrightarrow \Theta(i+k+1)^a[1] \ ,$$

and furthermore we have that $\mathrm{Hom}_{\mathcal{T}}(U, \Theta(i+k+1)[1]) = 0$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, \Theta(i+k+s)[1])$ to η, with $s \in [-k, 0]$, we get the exact sequence

$$(U_k, \Theta(i+k+s)[1]) \to (U, \Theta(i+k+s)[1]) \to (\Theta(i+k+1)^a, \Theta(i+k+s)[1]).$$

Since $\mathrm{Hom}_{\mathcal{T}}(\Theta(i+k+1)^a, \Theta(i+k+s)[1]) = 0 = \mathrm{Hom}_{\mathcal{T}}(U_k, \Theta(i+k+s)[1]) = 0$, it follows that $\mathrm{Hom}_{\mathcal{T}}(U, \Theta(i+k+s)[1]) = 0$ for any $s \in [-k, 0]$. Thus $\mathrm{Hom}_{\mathcal{T}}(U, \Theta(j)[1]) = 0$ for any $j \in [i, i+k+1]$. On the other hand, by 5.7 (a), we have that $\mathrm{Hom}_{\mathcal{T}}(\Theta(i+k+1)^a, U_k) = 0$ since $U_k \in \mathfrak{F}(\{\Theta(j) \mid j \in [i, i+k]\})$; also by 5.7 (c), we get that $\mathrm{Hom}_{\mathcal{T}}(\Theta(i+k+1)^a, U_k[-1]) = 0$. Thus, by 3.5, there exists a distinguished triangle

$$\eta' : \quad \Theta(i+k+1)^d \longrightarrow U_{k+1} \longrightarrow U_k \longrightarrow \Theta(i+k+1)^d[1]$$

with $d \leq a$ and U_{k+1} an indecomposable direct summand of U. By base change (see 2.1), we have the following commutative diagram

$$\begin{tikzcd}
 & \Theta(i)[-1] \arrow[r, equal] \arrow[d] & \Theta(i)[-1] \arrow[d] & \\
\Theta(i+k+1)^d \arrow[r] \arrow[d, equal] & V_{k+1} \arrow[r] \arrow[d, "\mu_{k+1}"] & V_k \arrow[r] \arrow[d, "\mu_k"] & \Theta(i+k+1)^d[1] \arrow[d, equal] \\
\Theta(i+k+1)^d \arrow[r] & U_{k+1} \arrow[r] \arrow[d] & U_k \arrow[r] \arrow[d] & \Theta(i+k+1)^d[1] \\
 & \Theta(i) \arrow[r, equal] & \Theta(i), &
\end{tikzcd}$$

where the rows and columns are distinguished triangles. Using the fact that $V_k \in \mathfrak{F}(\{\Theta(j) \mid i < j \leq i+k\})$, it follows by 4.2 that $V_{k+1} \in \mathfrak{F}(\{\Theta(j) \mid i < j \leq i+k+1\})$. Moreover $\mathrm{Hom}_{\mathcal{T}}(U_{k+1}, \Theta(j)[1]) = 0$ for $j \in [i, i+k+1]$, since

U_{k+1} is an indecomposable direct summand of U. Hence, the desired triangle is the first column of the preceding diagram, that is, ξ_{k+1} is the triangle

$$V_{k+1} \xrightarrow{\mu_{k+1}} U_{k+1} \longrightarrow \Theta(i) \longrightarrow V_{k+1}[1]\ .$$

□

Theorem 5.9. *Let $(\Theta, \leq)$ be a Θ-system of size t, in an artin triangulated R-category $\mathcal{T}$. Then, there exists a unique, up to isomorphism, family $\mathbf{Q}$ of objects in $\mathcal{T}$ such that $(\Theta, \mathbf{Q}, \leq)$ is a Θ-projective system of size t in $\mathcal{T}$.*

Proof. Without lost of generality, we can assume that $\leq$ is the natural order on the set $[1, t]$. For each $i < t$, we set $\eta_i := \xi_{t-i}$ where ξ_{t-i} is the distinguished triangle of 5.8

$$\xi_{t-i}\ :\ V_{t-i} \longrightarrow U_{t-i} \longrightarrow \Theta(i) \longrightarrow V_{t-i}[1].$$

Let $K(i) := V_{t-i}$ and $Q(i) := U_{t-i}$. Then, we have that $K(i) \in \mathfrak{F}(\{\Theta(j) \mid j > i\})$ and $\mathrm{Hom}_{\mathcal{T}}(Q(i), \Theta(j)[1]) = 0$ for $j \geq i$. From the triangle ξ_{t-i}, it follows that $Q(i) \in \mathfrak{F}(\{\Theta(j) \mid j \geq i\})$. By 5.7 (b) and (c), we conclude that $\mathrm{Hom}_{\mathcal{T}}(Q(i), \Theta(r)[1]) = 0$ for $r \leq i$ and $\mathrm{Hom}_{\mathcal{T}}(Q(i), \Theta(r)[-1]) = 0\ \forall\ r$. Therefore $Q(i) \in {}^{\perp}\Theta[-1] \cap {}^{\perp}\Theta[1]\ \ \forall i$. For $i = t$, we take the triangle η_t as follows

$$0 \longrightarrow \Theta(t) \xrightarrow{1} \Theta(t) \longrightarrow 0$$

and we set $Q(t) := \Theta(t)$ and $K(t) := 0$, so this triangle has the desired conditions. Finally, if there is another family $\mathbf{Q'}$ such that $(\Theta, \mathbf{Q'}, \leq)$ is a Θ-projective system of size t, then by 5.6 we get that $\mathbf{Q} \simeq \mathbf{Q'}$. □

Lemma 5.10. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in a triangulated R-category $\mathcal{T}$. Then, the R-functor $\mathrm{Hom}_{\mathcal{T}}(Q', -) : \mathfrak{F}(\Theta) \to \mathrm{mod}\,(R)$ is exact, for any $Q' \in \mathrm{add}\,(Q)$.*

Proof. Let $\eta\ :\ A \to B \to C \to A[1]$ be a distinguished triangle in $\mathfrak{F}(\Theta)$ and $Q' \in \mathrm{add}\,(Q)$. Applying $\mathrm{Hom}_{\mathcal{T}}(Q', -)$ to η, we get the exact sequence

$$(Q', C[-1]) \to (Q', A) \to (Q', B) \to (Q', C) \to (Q', A[1]).$$

Since $\mathbf{Q} \subseteq {}^{\perp}\Theta[-1] \cap {}^{\perp}\Theta[1]$, it follows from 4.3 that $\mathrm{Hom}_{\mathcal{T}}(Q', C[-1]) = \mathrm{Hom}_{\mathcal{T}}(Q', A[1]) = 0$. Thus, such a functor is exact. □

Proposition 5.11. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in a Hom-finite triangulated R-category $\mathcal{T}$. Then, the following statements hold.*

(a) *For any $M \in \mathfrak{F}(\Theta)$ the filtration multiplicity $[M : \Theta(i)]_{\xi}$ of $\Theta(i)$ in M doest not depend on the given Θ-filtration ξ of M and hence it will be denoted by $[M : \Theta(i)]$. In particular $\ell_{\Theta}(M) = \sum_{i=1}^{t}[M : \Theta(i)]$.*

(b) *$Q(i) \not\simeq Q(j)$ if $i \neq j$.*

Proof. (a) Consider a Θ-filtration ξ of $M \in \mathfrak{F}(\Theta)$

$$\xi = \{\xi_l \ : \ M_{l-1} \longrightarrow M_l \longrightarrow \Theta(j_l) \longrightarrow M_{l-1}[1]\ \}_{l=0}^{n},$$

where $M_{-1} = 0 = \Theta(j_0)$, $j_l \in [1,t]$ for $l \geq 1$, and $M_n = M$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(Q(i), -)$ to each triangle ξ_j, and by setting $\langle X, Y\rangle := \ell_R(\mathrm{Hom}_{\mathcal{T}}(X,Y))$, we get the following equalities

$$\begin{aligned}
\langle Q(i), M_1\rangle &= \langle Q(i), 0\rangle + \langle Q(i), \Theta(j_1)\rangle,\\
\langle Q(i), M_2\rangle &= \langle Q(i), \Theta(j_1)\rangle + \langle Q(i), \Theta(j_2)\rangle,\\
\langle Q(i), M_3\rangle &= \langle Q(i), M_2\rangle + \langle Q(i), \Theta(j_3)\rangle,\\
&\vdots\\
\langle Q(i), M\rangle &= \langle Q(i), M_{n-1}\rangle + \langle Q(i), \Theta(j_n)\rangle.
\end{aligned}$$

Let $c_i := \langle Q(i), M\rangle = \sum_{j=1}^{t}[M : \Theta(j)]_\xi \langle Q(i), \Theta(j)\rangle$. Consider the matrix $D := (d_{ij})$, where $d_{ij} := \langle Q(i), \Theta(j)\rangle$. By 5.5 (b), we have that D is an upper triangular matrix with $d_{ii} \neq 0 \ \ \forall\, i$, and thus $\det(D) \neq 0$. By using the column vectors $X := ([M : \Theta(1)]_\xi, [M : \Theta(2)]_\xi, \cdots, [M : \Theta(t)]_\xi)^t$ and $C := (c_1, c_2, \cdots, c_t)^t$, the above equalities can be written as a matrix equation $D \cdot X = C$. Since $\det(D) \neq 0$, we obtain that $X = D^{-1} \cdot C$, and hence $[M : \Theta(j)]_\xi$ only depends on the numbers $c_i = \langle Q(i), M\rangle$ and $d_{ij} = \langle Q(i), \Theta(j)\rangle$.

(b) Let $i \neq j$. We can assume that $j > i$. Then, by (a) and 5.2 (PS5), it follows that $[Q(i) : \Theta(i)] = 1$ and $[Q(j) : \Theta(i)] = 0$, and thus $Q(i) \not\simeq Q(j)$. $\square$

Definition 5.12. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in a Hom-finite triangulated R-category $\mathcal{T}$. The* **Θ-support** *of $M \in \mathfrak{F}(\Theta)$, is the set*

$$\mathrm{Supp}_\Theta(M) := \{i \in [1,t] \mid [M : \Theta(i)] \neq 0\}.$$

For $0 \neq M \in \mathfrak{F}(\Theta)$, let $\max(M)$ denote the maximum of $\mathrm{Supp}_\Theta(M)$ with respect to the linear order $\leq$, and similarly, $\min(M)$ denote the minimum of $\mathrm{Supp}_\Theta(M)$ with respect to the linear order $\leq$. Finally, we set $\max(0) := -\infty$ and $\min(0) := +\infty$.

Theorem 5.13. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in a Hom-finite triangulated R-category $\mathcal{T}$, and let $M \in \mathfrak{F}(\Theta)$ and $i := \min(M)$. Then, there exists a distinguished triangle in $\mathcal{T}$*

$$N \longrightarrow Q_0(M) \xrightarrow{\varepsilon_M} M \longrightarrow N[1]$$

satisfying the following conditions:

(a) $N \in \mathfrak{F}(\Theta)$ *and* $Q_0(M) \in \mathrm{add}\,(\bigoplus_{j\geq i} Q(j))$,

(b) $\min(M) < \min(N)$ *if* $M \neq 0$,

(c) $\varepsilon_M : Q_0(M) \to M$ *is a* $\mathcal{P}(\Theta)$*-precover of* M.

Proof. If $M = 0$, the zero distinguished triangle $0 \to 0 \to 0 \to 0$ is the desired one. Without lost of generality, it can be assumed that $\leq$ is the natural order on the set $[1, t]$.

Let $M \neq 0$. Then, by 5.5 (a) and 4.7 (c), there is a distinguished triangle

$$N \xrightarrow{\varphi} M \xrightarrow{\psi} \Theta(i)^{m_i} \longrightarrow N[1]$$

with $N \in \mathfrak{F}(\Theta)$ and $\min(M) < \min(N)$. We proceed by reverse induction on $i = \min(M)$. If $i = \min(M) = t$, we have that $N = 0$ and hence the desired triangle is $0 \to \Theta(t)^{m_i} \to \Theta(t)^{m_i} \to 0$, since $Q(t) \simeq \Theta(t)$.
Let $i = \min(M) < t$. If $N = 0$, we have that $M = \Theta(i)^{m_i}$ and thus the following distinguished triangle (see 5.2 (PS5)) is the desired one

$$K(i)^{m_i} \longrightarrow Q(i)^{m_i} \xrightarrow{\beta_i^{m_i}} \Theta(i)^{m_i} \longrightarrow K(i)^{m_i}[1].$$

Suppose that $N \neq 0$. Since $i = \min(M) < \min(N)$, by induction, there is a distinguished triangle

$$N' \longrightarrow Q_0(N) \xrightarrow{\varepsilon_N} N \longrightarrow N'[1]$$

such that $i < \min(N) < \min(N') =: i'$, $Q_0(N) \in \operatorname{add}(\bigoplus_{j \geq i'} Q(j))$ and $\varepsilon_N : Q_0(N) \to N$ is a $\mathcal{P}(\Theta)$-precover of N. By base change (see 2.1), we obtain the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
 & K(i)^{m_i} \arrow[r, equal] \arrow[d] & K(i)^{m_i} \arrow[d] & \\
N \arrow[r, "i_1"] \arrow[d, equal] & E \arrow[r, "p_2"] \arrow[d, "\theta"] & Q(i)^{m_i} \arrow[r] \arrow[d, "\beta_i^{m_i}"] & N[1] \arrow[d, equal] \\
N \arrow[r, "\varphi"] & M \arrow[r, "\psi"] \arrow[d] & \Theta(i)^{m_i} \arrow[r] \arrow[d] & N[1] \\
 & K(i)^{m_i}[1] \arrow[r, equal] & K(i)^{m_i}[1]. &
\end{tikzcd}$$

Since $N \in \mathfrak{F}(\Theta)$, we have that $\operatorname{Hom}_{\mathcal{T}}(Q(i), N[1]) = 0$. Thus, the first row, in the diagram above, splits. So, there is $i_2 : Q(i)^{m_i} \to E$ such that $\beta_i^{m_i} = \psi\theta i_2$. Define $\alpha := \theta i_2$ and $\varepsilon := (\varphi\varepsilon_N, \alpha) : Q_0(N) \bigoplus Q(i)^{m_i} \longrightarrow M$. Hence, we get the following commutative diagram

$$\begin{tikzcd}
Q_0(N) \arrow[r, "j_1"] \arrow[d, "\varepsilon_N"] & Q_0(M) \arrow[r, "\pi_2"] \arrow[d, "\varepsilon"] & Q(i)^{m_i} \arrow[r, "0"] \arrow[d, "\beta_i^{m_i}"] & Q_0(N)[1] \arrow[d, "\varepsilon_N[1]"] \\
N \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & \Theta(i)^{m_i} \arrow[r] & N[1]
\end{tikzcd}$$

with $Q_0(M) := Q_0(N) \bigoplus Q(i)^{m_i}$, where the rows are distinguished triangles, $j_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\pi_2 := \begin{pmatrix} 0 & 1 \end{pmatrix}$. Since $\mathbf{Q} \subseteq {}^{\perp}\Theta[-1]$ and $Q_0(N) \in \operatorname{add}(\bigoplus_{j \geq i'} Q(j))$, we conclude that $\operatorname{Hom}_{\mathcal{T}}(Q_0(N), \Theta(i)^{m_i}[-1]) = 0$. Thus, by 2.3, we obtain the following diagram in $\mathcal{T}$

$$\begin{tikzcd}
N' \arrow[r] \arrow[d] & P[-1] \arrow[r] \arrow[d] & K(i)^{m_i} \arrow[r] \arrow[d] & N'[1] \arrow[d] \\
Q_0(N) \arrow[r, "j_1"] \arrow[d, "\varepsilon_N"] & Q_0(M) \arrow[r, "\pi_2"] \arrow[d, "\varepsilon"] & Q(i)^{m_i} \arrow[r] \arrow[d, "\beta_i^{m_i}"] & Q_0(N)[1] \arrow[d, "\varepsilon_N[1]"] \\
N \arrow[r, "\varphi"] \arrow[d] & M \arrow[r, "\psi"] \arrow[d, "\gamma"] & \Theta(i)^{m_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "IX"] & N[1] \arrow[d] \\
N'[1] \arrow[r] & P \arrow[r] & K(i)^{m_i}[1] \arrow[r] & N'[2].
\end{tikzcd}$$

where the rows and columns, in the diagram above, are distinguished triangles and all squares commute, except the one marked with IX, which anticommutes. We claim that the following distinguished triangle in $\mathcal{T}$

$$P[-1] \longrightarrow Q_0(M) \xrightarrow{\varepsilon} M \longrightarrow P$$

is the desired one. Indeed, we have that $Q_0(M) \in \operatorname{add}(\bigoplus_{j \geq i} Q(j))$ since $Q_0(N) \in \operatorname{add}(\bigoplus_{j \geq i'} Q(j))$ with $i < \min(N) < \min(N') = i'$ and $Q(i)^{m_i} \in \mathfrak{F}(\{\Theta(j) \mid j \geq i\})$. By considering the first row from the last diagram, it follows that $P[-1] \in \mathfrak{F}(\{\Theta(j) \mid j > i\})$ since $K(i)^{m_i} \in \mathfrak{F}(\{\Theta(j) \mid j > i\})$ and $N' \in \mathfrak{F}(\{\Theta(j) \mid j > i'\})$ with $i < i'$. Therefore $i = \min(M) < \min(P[-1])$.
Finally, we show that ε is a $\mathcal{P}(\Theta)$-precover of M. Indeed, let $h : X \to M$ be a morphism in $\mathcal{T}$ with $X \in \mathcal{P}(\Theta)$, and consider the morphism $\gamma h : X \to P$. Since $P[-1] \in \mathfrak{F}(\Theta)$, we have that $\gamma h = 0$ and so there is $h' : X \to Q_0(M)$ such that $h = \varepsilon h'$. Then, the morphism ε is a $\mathcal{P}(\Theta)$-precover of M, proving the result. $\square$

Corollary 5.14. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in a Hom-finite triangulated R-category $\mathcal{T}$. Then*

$$\operatorname{add}(Q) = \mathfrak{F}(\Theta) \cap \mathcal{P}(\Theta).$$

Proof. It is clear that $\operatorname{add}(Q) \subseteq \mathfrak{F}(\Theta) \cap \mathcal{P}(\Theta)$. Let $M \in \mathfrak{F}(\Theta) \cap \mathcal{P}(\Theta)$. Then, by 5.13, there is a distinguished triangle in $\mathcal{T}$

$$\eta : \quad N \longrightarrow Q_0(M) \xrightarrow{\varepsilon_M} M \longrightarrow N[1],$$

where $Q_0(M) \in \operatorname{add}(Q)$ and $N \in \mathfrak{F}(\Theta)$. Thus, the triangle η splits and then $M \in \operatorname{add}(Q)$; proving the result. $\square$

6. The standardly stratified algebra associated to a Θ-projective system

Theorem 6.1. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in an artin triangulated R-category $\mathcal{T}$, and let $A := \mathrm{End}_{\mathcal{T}}(Q)^{op}$, $e_Q := \mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathcal{T} \to \mathrm{mod}\,(A)$ and ${}_AP(i) := e_Q(Q(i))$ for each $i \in [1,t]$. Then, the following statements holds.*

(a) *The family ${}_AP := \{{}_AP(i) \mid i \in [1,t]\}$ is a representative set of the indecomposable projective A-modules. In particular, A is basic and $rk\, K_0(A) = t$.*

(b) $e_Q(\Theta(i)) \simeq {}_A\Delta(i) \quad \forall\, i \in [1,t]$, *where ${}_A\Delta$ is computed by using ${}_AP$ and the given order $\leq$ on $[1,t]$.*

(c) $(A, \leq)$ *is a standardly stratified algebra, that is,* $\mathrm{proj}\,(A) \subseteq \mathfrak{F}({}_A\Delta)$.

(d) *The restriction $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is an exact equivalence of R-categories.*

Proof. By 5.10, we know that $e_Q = \mathrm{Hom}_{\mathcal{T}}(Q, -)|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathrm{mod}\,(A)$ is an exact functor.

(a) It follows by 3.3 and 5.11 (b).

(b) and (c) Let $i \in [1,t]$. By 5.2 (PS5) and 5.13, we have two distinguished triangles in $\mathcal{T}$

$$\eta_i \;:\; K(i) \xrightarrow{\alpha_i} Q(i) \longrightarrow \Theta(i) \longrightarrow K(i)[1],$$

$$\eta_i' \;:\; K' \longrightarrow Q' \xrightarrow{\lambda_i} K(i) \longrightarrow K'[1],$$

where $K(i), K' \in \mathfrak{F}(\{\Theta(j) \mid j > i\}$ and $Q' \in \mathrm{add}\,(\oplus_{j>i}\, Q(j))$. Applying the functor $e_Q = \mathrm{Hom}_{\mathcal{T}}(Q, -)$ to the triangles η_i and η_i', we get the following exact sequence in $\mathrm{mod}\,(A)$

$$\varepsilon_i \;:\; e_Q(Q') \xrightarrow{e_Q(\gamma_i)} {}_AP(i) \longrightarrow e_Q(\Theta(i)) \longrightarrow 0,$$

where $\gamma_i := \alpha_i \lambda_i$. We assert that

$$\mathrm{Im}\,(e_Q(\gamma_i)) = \mathrm{Tr}_{\oplus_{j>i}\ {}_AP(j)}\,({}_AP(i)).$$

Indeed, using the fact that $e_Q(Q') \in \mathrm{add}\,(\oplus_{j>i}\ {}_AP(j))$, it follows that $\mathrm{Im}\,(e_Q(\gamma_i)) \subseteq \mathrm{Tr}_{\oplus_{j>i}\ {}_AP(j)}\,({}_AP(i))$. In order to see the other inclusion, let $j > i$ and consider a morphism $f : {}_AP(j) \to {}_AP(i)$. By 5.2 (PS3) and 3.3 (c), we conclude that $\mathrm{Hom}_A({}_AP(j), e_Q(\Theta(i)) = 0$ and hence f factorizes trough $e_Q(\gamma_i)$; proving our assertion. Finally, by this assertion and the exact sequence ε_i, we obtain (b) and (c).

(d) Since $e_Q : \mathfrak{F}(\Theta) \to \mathrm{mod}\,(A)$ is an exact functor, it remains to prove that $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is full, faithful and a dense functor.

Let $M \in \mathfrak{F}(\Theta)$. We prove, by induction on $\ell_\Theta(M)$, that $e_Q(M) \in \mathfrak{F}({}_A\Delta)$. If $\ell_\Theta(M) \leq 1$, then $M = \Theta(i)^{m_i}$ for some i, and hence by (a) it follows that

$e_Q(M) \in \mathfrak{F}({}_A\Delta)$.
Let $\ell_\Theta(M) > 1$. Then, from 5.5 (a) and 4.7 (c), there is a distinguished triangle $N \to M \to \Theta(i)^m \to N[1]$ in $\mathfrak{F}(\Theta)$ such that $\ell_\Theta(N) < \ell_\Theta(M)$. Therefore, by induction and since $\mathfrak{F}({}_A\Delta)$ is closed under extensions, we conclude that $e_Q(M)$ belongs to $\mathfrak{F}({}_A\Delta)$; proving that $\mathrm{Im}\,(e_Q|_{\mathfrak{F}(\Theta)}) \subseteq \mathfrak{F}({}_A\Delta)$.

Now, we prove that $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is full and faithful. Indeed, let $M, N \in \mathfrak{F}(\Theta)$. By 5.13 and the exactness of the functor e_Q, we get an exact sequence $\varepsilon \; : \; e_Q(Q_1) \to e_Q(Q_0) \to M \to 0$ in $\mathrm{mod}\,(A)$ such that $Q_0, Q_1 \in \mathrm{add}\, Q$. From ε, we obtain the following exact and commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & {}_{\mathcal{T}}(M,N) & \longrightarrow & {}_{\mathcal{T}}(Q_0,N) & \longrightarrow & {}_{\mathcal{T}}(Q_1,N) \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\
0 & \longrightarrow & {}_A(e_Q(M), e_Q(N)) & \longrightarrow & {}_A(e_Q(Q_0), e_Q(N)) & \longrightarrow & {}_A(e_Q(Q_1), e_Q(N))
\end{array}$$

where α_2 and α_3 are isomorphism (see 3.3 (c)). Thus, by using the so-called Five's Lemma, it follows that α_1 is an isomorphism; proving that e_Q is full and faithful.

Finally, we see that $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is dense. Indeed, let $M \in \mathfrak{F}({}_A\Delta)$. We proceed by induction on the ${}_A\Delta$-length $\ell_{{}_A\Delta}(M)$. If $\ell_{{}_A\Delta}(M) = 1$ then $M \simeq {}_A\Delta(i) \simeq e_Q(\Theta(i))$ for some i.
Let $\ell_{{}_A\Delta}(M) > 1$. Then, there is an exact sequence in $\mathrm{mod}\,(A)$

$$0 \longrightarrow {}_A\Delta(i) \longrightarrow M \longrightarrow M/{}_A\Delta(i) \longrightarrow 0,$$

where $\ell_{{}_A\Delta}(M/{}_A\Delta(i)) = \ell_{{}_A\Delta}(M) - 1$ for some i. So, by induction, there exists $Z \in \mathfrak{F}(\Theta)$ such that $e_Q(Z) \simeq M/{}_A\Delta(i)$. Moreover, by 5.13, there is a distinguished triangle $\eta_Z \; : \; Z' \overset{u}{\to} Q_0(Z) \overset{\varepsilon_Z}{\to} Z \to Z'[1]$ in $\mathfrak{F}(\Theta)$, with $Q_0(Z) \in \mathrm{add}\,(Q)$; and thus, we get the following exact and commutative

diagram in $\mathrm{mod}\,(A)$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & e_Q(Z') \arrow[r, equal] \arrow[d, "\mu"] & e_Q(Z') \arrow[d, "e_Q(u)"] & \\
\eta \,:\, 0 \arrow[r] & {}_A\Delta(i) \arrow[r, "i_1"] \arrow[d, equal] & C \arrow[r, "p_2"] \arrow[d, "\lambda"] & e_Q(Q_0(Z)) \arrow[r] \arrow[d, "e_Q(\varepsilon_Z)"] & 0 \\
0 \arrow[r] & {}_A\Delta(i) \arrow[r] & M \arrow[r] \arrow[d] & e_Q(Z) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}
$$

Since $e_Q(Q_0(Z)) \in \mathrm{proj}\,(A)$, the exact sequence η splits and hence $C = {}_A\Delta(i) \bigoplus e_Q(Q_0(Z)) \simeq e_Q(\Theta(i) \bigoplus Q_0(Z))$, $i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $p_2 = (0,1)$. That is $\mu = \begin{pmatrix} \varphi \\ e_Q(u) \end{pmatrix}$ with $\varphi : e_Q(Z') \to e_Q(\Theta(i))$. Since the restriction $e_Q\,|_{\mathfrak{F}(\Theta)}$ is full, there exists $h : Z' \to \Theta(i)$ such that $e_Q(h) = \varphi$ and hence $\mu = e_Q(\psi)$, where $\psi := \begin{pmatrix} h \\ u \end{pmatrix}$. Then, by completing ψ to a distinguished triangle and from 2.3, we get the following commutative diagram

$$
\begin{tikzcd}
 & \Theta(i) \arrow[r, equal] \arrow[d] & \Theta(i) \arrow[d] & \\
Z' \arrow[r, "\psi"] \arrow[d, equal] & \Theta(i) \bigoplus Q_0(Z) \arrow[r] \arrow[d, "\pi_2"] & X \arrow[r] \arrow[d, "\alpha"] & Z'[1] \arrow[d, equal] \\
Z' \arrow[r, "u"] & Q_0(Z) \arrow[r, "\varepsilon_Z"] \arrow[d] & Z \arrow[r] \arrow[d] & Z'[1] \\
 & \Theta(i)[1] \arrow[r, equal] & \Theta(i)[1], &
\end{tikzcd}
$$

where the rows and columns are distinguished triangles and $\pi_2 := (0,1)$. Observe that $X \in \mathfrak{F}(\Theta)$ since $\mathfrak{F}(\Theta)$ is closed under extensions. Thus, by applying e_Q to the first row, in the diagram above, we get the exact sequence

$$0 \longrightarrow e_Q(Z') \xrightarrow{e_Q(\psi)} e_Q(\Theta(i) \bigoplus Q_0(Z)) \longrightarrow e_Q(X) \longrightarrow 0.$$

But $e_Q(\psi) = \mu$ and hence $e_Q(X) \simeq \mathrm{Coker}\,(\mu) = M$; proving that $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is dense. $\square$

Proposition 6.2. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in an artin triangulated R-category $\mathcal{T}$. Then, the following statements hold.*

(a) *$\mathfrak{F}(\Theta)$ is closed under extensions and direct summands.*
(b) *$\Theta(i)$ is indecomposable for each $i \in [1, t]$.*
(c) *For any object $M \in \mathfrak{F}(\Theta)$, there exists a distinguished triangle $Z \to Q_M \to M \to Z[1]$ in $\mathfrak{F}(\Theta)$ such that $Q_M \to M$ is an $\mathrm{add}\,(Q)$-cover of M, and $\min\,(M) < \min\,(Z)$ if $M \neq 0$.*

Proof. Let $A := \mathrm{End}_{\mathcal{T}}(Q)^{op}$. We know by 6.1 that: $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is an exact equivalence, $(A, \leq)$ is an standardly stratified algebra and $e_Q(\Theta(i)) \simeq {}_A\Delta(i)\ \ \forall\, i$. Since ${}_A\Delta(i)$ is indecomposable and $\mathrm{End}_{\mathcal{T}}(\Theta(i)) \simeq \mathrm{End}_A({}_A\Delta(i))$, it follows (b). To prove (a), we use the well-known fact that $\mathfrak{F}({}_A\Delta)$ is closed under direct summands (see [2]). Indeed, this property can be carried back to $\mathfrak{F}(\Theta)$ by using the equivalence e_Q and that both $\mathcal{T}$ and $\mathrm{mod}\,(A)$ are Krull-Schmidt categories.

Finally, since $\mathfrak{F}({}_A\Delta)$ is a resolving subcategory of $\mathrm{mod}\,(A)$ (see [2]), by using 5.13 and the exact equivalence $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$, we get (c). $\square$

Corollary 6.3. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in an artin triangulated R-category $\mathcal{T}$, and let $\mathbf{K} := \{K(i)\}_{i=1}^{t}$ where, for each i, $K(i)$ is the object appearing in 5.2 (PS5). If $\mathrm{Hom}_{\mathcal{T}}(\mathbf{K}[2], \Theta) = 0$ then $(\Theta, \leq)$ is a Θ-system of size t, in $\mathcal{T}$.*

Proof. It follows from 6.2 (b) and 5.5 (a), (c). $\square$

Corollary 6.4. *Let $(\Theta, \leq)$ be a Θ-system of size t, in an artin triangulated R-category $\mathcal{T}$. Then, the following statements hold.*

(a) *$\mathfrak{F}(\Theta)$ is closed under extensions and direct summands.*
(b) *There is a unique, up to isomorphism, Θ-projective system $(\Theta, \mathbf{Q}, \leq)$ of size t, which is associated to the Θ-system $(\Theta, \leq)$.*
(c) *For any object $M \in \mathfrak{F}(\Theta)$, there exists a distinguished triangle $Z \to Q_M \to M \to Z[1]$ in $\mathfrak{F}(\Theta)$ such that $Q_M \to M$ is an $\mathrm{add}\,(Q)$-cover of M, and $\min\,(M) < \min\,(Z)$ if $M \neq 0$.*
(d) $\mathfrak{F}(\Theta) \cap \mathcal{P}(\Theta) = \mathrm{add}\,(Q)$.

Proof. It follows from 5.9, 6.2 and 5.14. $\square$

The previous results can be seen also under the light of the so-called cotorsion pairs in the sense of Iyama-Nakaoka-Yoshino (see [17] and [26]). Such cotorsion pairs are studied extensively in relation with cluster tilting categories, t-structures and co-t-structures.

Definition 6.5. *A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories in a triangulated category $\mathcal{T}$ is called a cotorsion pair if the following conditions hold.*

(a) *$\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands in $\mathcal{T}$.*
(b) *$\operatorname{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y}) = 0$ and $\mathcal{T} = \mathcal{X} * \mathcal{Y}[1]$.*

The core of the cotorsion pair $(\mathcal{X},\mathcal{Y})$ is the subcategory $\mathcal{X} \cap \mathcal{Y}$.

Corollary 6.6. *Let $(\Theta,\leq)$ be a Θ-system of size t, in an artin triangulated R-category $\mathcal{T}$. Then, the pairs $(\mathcal{P}(\Theta),\mathfrak{F}(\Theta))$ and $(\mathfrak{F}(\Theta),\mathcal{I}(\Theta))$ are cotorsion pairs.*

Proof. Since $\mathcal{P}(\Theta) := {}^{\perp}\mathfrak{F}(\Theta)[1]$ and $\mathcal{I}(\Theta) := \mathfrak{F}(\Theta)^{\perp}[-1]$, it follows that these classes are closed under direct summands in $\mathcal{T}$. Furthermore, by 6.4 (a), we also know that $\mathfrak{F}(\Theta)$ is closed under direct summands in $\mathcal{T}$. Finally, from 4.14, we get that $\mathcal{P}(\Theta) * \mathfrak{F}(\Theta)[1] = \mathcal{T} = \mathfrak{F}(\Theta) * \mathcal{I}(\Theta)[1]$. $\square$

Remark 6.7. *Let $(\Theta,\leq)$ be a Θ-system of size t, in an artin triangulated R-category $\mathcal{T}$. Observe that, by 5.4, 6.1 and 6.4, the cotorsion pairs $(\mathcal{P}(\Theta),\mathfrak{F}(\Theta))$ and $(\mathfrak{F}(\Theta),\mathcal{I}(\Theta))$ have the following properties. Their cores are determined by, respectively, the Θ-projective system $(\Theta,\mathbf{Q},\leq)$ and the Θ-injective system $(\Theta,\mathbf{Y},\leq)$ as follows:*

$$\mathfrak{F}(\Theta) \cap \mathcal{P}(\Theta) = \operatorname{add}(Q) \quad \textit{and} \quad \mathfrak{F}(\Theta) \cap \mathcal{I}(\Theta) = \operatorname{add}(Y).$$

Moreover, the endomorphism algebras $\operatorname{End}_{\mathcal{T}}(Q)^{op}$ and $\operatorname{End}_{\mathcal{T}}(Y)$ are standardly stratified algebras.

7. The bounded derived category $\mathrm{D}^b(\mathfrak{F}(\Theta))$

We recall that an exact category is and additive category $\mathcal{A}$ endowed with a class $\mathcal{E}$ of pairs $M \xrightarrow{i} E \xrightarrow{p} N$ in $\mathcal{A}$ closed under isomorphisms and satisfying a list of axioms [29, 21]. An exact category $(\mathcal{A},\mathcal{E})$ is saturated if every idempotent in $\mathcal{A}$ splits and so, in this case (see, for example in [22, 27]), there exists the bounded derived category $\mathrm{D}^b(\mathcal{A})$.

Let $(A,\leq)$ be an standardly stratified algebra. It is well-known that $\mathfrak{F}({}_A\Delta)$ is an additive category, which is closed under extensions and every idempotent in $\mathfrak{F}({}_A\Delta)$ splits. Consider the class $\operatorname{Ex}({}_A\Delta)$ of all pairs $M \xrightarrow{i} E \xrightarrow{p} N$ in $\mathfrak{F}({}_A\Delta)$ such that $0 \to M \xrightarrow{i} E \xrightarrow{p} N \to 0$ is an exact sequence in $\operatorname{mod}(A)$. Then, the pair $(\mathfrak{F}({}_A\Delta), \operatorname{Ex}({}_A\Delta))$ is an exact category, and since it is saturated, there exists the bounded derived category $\mathrm{D}^b(\mathfrak{F}({}_A\Delta))$. We denote by $\mathrm{D}^b(A)$ to the bounded derived category of the abelian category $\operatorname{mod}(A)$.

Lemma 7.1. *Let $(A,\leq)$ be an standardly stratified algebra. Then*

$$\mathrm{D}^b(\mathfrak{F}({}_A\Delta)) \simeq \mathrm{D}^b(A)$$

as triangulated categories.

Proof. Since $(A,\leq)$ is an standardly stratified algebra, it follows by [2] that $\mathfrak{F}({}_A\Delta)$ is a resolving subcategory of $\operatorname{mod}(A)$. In particular, for any $M \in \mathfrak{F}({}_A\Delta)$, there is an exact sequence $0 \to M' \to P_0(M) \to M \to 0$ lying in

$\mathfrak{F}({}_A\Delta)$ and such that $P_0(M) \to M$ is the projective cover of M. Hence, the construction outlined in [5, Section 2] give us an equivalence

$$R_{>-\infty} : \mathrm{D}^b(\mathfrak{F}({}_A\Delta)) \to K^{-,b}(\mathrm{proj}\,(A))$$

as triangulated categories, where $R_{>-\infty} := \cdots R_{-2}R_{-1}R_0R_{>0}$. So, the lemma follows, since $K^{-,b}(\mathrm{proj}\,(A)) \simeq \mathrm{D}^b(A)$ as triangulated categories. □

Definition 7.2. *Let Θ be a class of objects in a triangulated category $\mathcal{T}$. We denote by $\mathrm{Ex}\,(\Theta)$ to the class of all the pairs $M \xrightarrow{i} E \xrightarrow{p} N$ in $\mathfrak{F}(\Theta)$ admitting a morphism $q : N \to M[1]$ such that $M \xrightarrow{i} E \xrightarrow{p} N \xrightarrow{q} M[1]$ is a distinguished triangle in $\mathcal{T}$.*

Theorem 7.3. *Let $(\Theta, \mathbf{Q}, \leq)$ be a Θ-projective system of size t, in an artin triangulated R-category $\mathcal{T}$. Consider $A := \mathrm{End}_{\mathcal{T}}(Q)^{op}$ and the functor $e_Q := \mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathcal{T} \to \mathrm{mod}\,(A)$, where $Q := \oplus_{i=1}^{t} Q(i)$. Then, the following statements hold.*

(a) *The pair $(\mathfrak{F}(\Theta), \mathrm{Ex}\,(\Theta))$ is an exact and Krull-Schmidt category. Moreover, the equivalence $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ satisfies that $e_Q(\mathrm{Ex}\,(\Theta)) = \mathrm{Ex}\,({}_A\Delta)$.*
(b) *The derived functor $R\mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathrm{D}^b(\mathfrak{F}(\Theta)) \to \mathrm{D}^b(A)$ is an equivalence of triangulated categories.*
(c) *$\mathrm{D}^b(\mathfrak{F}(\Theta)) \simeq K^{-,b}(\mathrm{add}\,(Q))$ as triangulated categories.*

Proof. (a) By 4.2 and 6.2 (a), we know that $\mathfrak{F}(\Theta)$ is closed under extensions and direct summands in the artin triangulated R-category $\mathcal{T}$. Thus $\mathfrak{F}(\Theta)$ is an additive and Krull-Schmidt category. Consider the class $\mathrm{Ex}_Q\,(\Theta)$ of all the pairs $M \xrightarrow{i} E \xrightarrow{p} N$ in $\mathfrak{F}(\Theta)$ satisfying that $e_Q(M) \xrightarrow{e_Q(i)} e_Q(E) \xrightarrow{e_Q(p)} e_Q(N)$ belongs to $\mathrm{Ex}\,({}_A\Delta)$. Since $e_Q : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_A\Delta)$ is an exact equivalence of R-categories (see 6.1) and $\mathfrak{F}({}_A\Delta)$ is closed under extensions, it follows that $\mathrm{Ex}\,(\Theta) \subseteq \mathrm{Ex}_Q\,(\Theta)$ and also that the pair $(\mathfrak{F}(\Theta), \mathrm{Ex}_Q\,(\Theta))$ is an exact category. It remains to see that $\mathrm{Ex}_Q\,(\Theta) \subseteq \mathrm{Ex}\,(\Theta)$.

Let $M \xrightarrow{i} E \xrightarrow{p} N$ be in $\mathrm{Ex}_Q\,(\Theta)$. Consider a distinguished triangle of the form $\eta\ :\ M \xrightarrow{i} E \xrightarrow{\lambda} C \xrightarrow{\omega} M[1]$. We assert that $e_Q(\lambda) : e_Q(E) \to e_Q(C)$ is surjective. In order to see that, we apply the functor e_Q to the triangle η; and then we get the exact sequence $e_Q(E) \xrightarrow{e_Q(\lambda)} e_Q(C) \xrightarrow{e_Q(\omega)} e_Q(M[1])$. But $e_Q(M[1]) = 0$ (see 5.2 (PS4)) and so $e_Q(\lambda) : e_Q(E) \to e_Q(C)$ is surjective. On the other hand, since $e_Q(p)e_Q(i) = 0$, it follows that $pi = 0$ and hence there exists $p' : C \to N$ such that $p'\lambda = p$. Therefore, we get the following

exact en commutative diagram in mod (A)

$$\begin{tikzcd}
0 \arrow[r] & e_Q(M) \arrow[r, "e_Q(i)"] \arrow[d, equal] & e_Q(E) \arrow[r, "e_Q(\lambda)"] \arrow[d, equal] & e_Q(C) \arrow[r] \arrow[d, "e_Q(p')"] & 0 \\
0 \arrow[r] & e_Q(M) \arrow[r, "e_Q(i)"] & e_Q(E) \arrow[r, "e_Q(p)"] & e_Q(N) \arrow[r] & 0.
\end{tikzcd}$$

Hence $e_Q(p')$ is an isomorphism in mod (A) and then $p' : C \to N$ is an isomorphism in $\mathcal{T}$. So the triangle $M \xrightarrow{i} E \xrightarrow{p} N \xrightarrow{\omega(p')^{-1}} M[1]$ is isomorphic to the distinguished triangle η; proving that $M \xrightarrow{i} E \xrightarrow{p} N$ belongs to $\mathrm{Ex}\,(\Theta)$. Thus $\mathrm{Ex}_Q\,(\Theta) = \mathrm{Ex}\,(\Theta)$.

(b) By (a), it follows that the derived functor

$$R\mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathrm{D}^b(\mathfrak{F}(\Theta)) \to \mathrm{D}^b(\mathfrak{F}({}_A\Delta))$$

is an equivalence of triangulated categories. Hence, (b) is a consequence of 7.1.

(c) By 3.3 (b) and 5.10, we get that $e_Q : \mathrm{add}\,(Q) \to \mathrm{proj}\,(A)$ is an exact equivalence of R-categories. Hence, we have that $K^{-,b}(\mathrm{add}\,(Q)) \simeq K^{-,b}(\mathrm{proj}\,(A))$ as triangulated categories. Therefore (c) follows from (b), since $K^{-,b}(\mathrm{proj}\,(A)) \simeq \mathrm{D}^b(A)$ as triangulated categories. □

Theorem 7.4. *Let $(\Theta, \leq)$ be a Θ-system, of size t, in an artin triangulated R-category $\mathcal{T}$. Then, the following statements hold.*

(a) *The pair $(\mathfrak{F}(\Theta), \mathrm{Ex}\,(\Theta))$ is an exact and Krull-Schmidt category.*
(b) *There exist a unique, up to isomorphism, families $\mathbf{Q}$ and $\mathbf{Y}$ of objects in $\mathcal{T}$, such that $(\Theta, \mathbf{Q}, \leq)$ is a Θ-projective system and $(\Theta, \mathbf{Y}, \leq)$ is a Θ-injective system.*
(c) *For the R-algebras $A := \mathrm{End}_{\mathcal{T}}(Q)^{op}$ and $B := \mathrm{End}_{\mathcal{T}}(Y)$, the derived functors*

$$R\mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathrm{D}^b(\mathfrak{F}(\Theta)) \to \mathrm{D}^b(A) \quad \textit{and} \quad R\mathrm{Hom}_{\mathcal{T}}(-, Y) : \mathrm{D}^b(\mathfrak{F}(\Theta)) \to \mathrm{D}^b(B)$$

are equivalences as triangulated categories.

(d) *Both pairs $(A, \leq)$ and $(B, \leq^{op})$ are standardly stratified algebras, and moreover, the algebras A and B are derived equivalent.*

Proof. Since the pair $(\Theta, \leq)$ is a Θ-system, of size t, in an artin triangulated R-category $\mathcal{T}$, it follows from 5.9 and its dual that (b) is true. Therefore, by 7.3 and its dual, we get (a) and (c). The fact that both pairs $(A, \leq)$ and $(B, \leq^{op})$ are standardly stratified algebras, can be obtained from 6.1 (c) and its dual. Finally, the fact that $\mathrm{D}^b(A) \simeq \mathrm{D}^b(B)$ as triangulated categories (see (c)) say us that A and B are derived equivalent. □

8. Examples

8.1. **From stratifying systems in module categories.** Let Λ be an artin R-algebra and let $\mathrm{D}^b(\Lambda)$ be the bounded derived category of complexes in $\mathrm{mod}\,(\Lambda)$. It is well-known that the canonical functor $\imath_0 : \mathrm{mod}\,(\Lambda) \to \mathrm{D}^b(\Lambda)$, which sends $M \in \mathrm{mod}\,(\Lambda)$ to the stalk complex $M[0]$ concentrated in degree zero, is additive full and faithful. Hence, through the functor $\imath_0$, the module category $\mathrm{mod}\,(\Lambda)$ can be considered as a full additive subcategory of $\mathrm{D}^b(\Lambda)$. Furthermore $\mathrm{Ext}^k_\Lambda(X, Y) \simeq \mathrm{Hom}_{\mathrm{D}^b(\Lambda)}(X[0], Y[k])$ for any $k \in \mathbb{Z}$ and $X, Y \in \mathrm{mod}\,(\Lambda)$.

In what follows, we recall from [23] the notion of stratifying systems; for a further development of such systems, see in [16, 23, 24, 25].

Definition 8.1. [23] *A stratifying system* $(\Theta, \leq)$, *of size* t *in* $\mathrm{mod}\,(\Lambda)$ *consist of the following data.*

(SS1) $\leq$ *is a linear order on* $[1, t]$.
(SS2) $\Theta = \{\Theta(i)\}_{i=1}^t$ *is a family of indecomposable objects in* $\mathrm{mod}\,(\Lambda)$.
(SS3) $\mathrm{Hom}_\Lambda(\Theta(j), \Theta(i)) = 0$ *for* $j > i$.
(SS4) $\mathrm{Ext}^1_\Lambda(\Theta(j), \Theta(i)) = 0$ *for* $j \geq i$.

By using the formula $\mathrm{Ext}^k_\Lambda(X, Y) \simeq \mathrm{Hom}_{\mathrm{D}^b(\Lambda)}(X[0], Y[k])$, we get that any stratifying system $(\Theta, \leq)$, of size t in $\mathrm{mod}\,(\Lambda)$, produces the $\Theta[0]$-system $(\Theta[0], \leq)$ of size t in the triangulated category $\mathrm{D}^b(\Lambda)$.

8.2. **Exceptional sequences.** The notion of exceptional sequence originates from the study of vector bundles (see, for instance, [7, 19]). Here, $\mathcal{T}$ denotes an artin triangulated R-category.

Definition 8.2. [7, 32] *An exceptional sequence of size* t, *in the triangulated category* $\mathcal{T}$, *is a sequence* $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ *of objects in* $\mathcal{T}$ *satisfying the following conditions.*

(ES1) $\mathrm{End}_\mathcal{T}(\mathcal{E}_i)$ *is a division ring, for each* $i \in [1, t]$.
(ES2) $\mathrm{Hom}_\mathcal{T}(\mathcal{E}_i, \mathcal{E}_i[k]) = 0\ \forall\, i \in [1, t],\ \forall\, k \in \mathbb{Z} - \{0\}$.
(ES3) $\mathrm{Hom}_\mathcal{T}(\mathcal{E}_j, \mathcal{E}_i[k]) = 0$ *for* $j > i$ *and* $\forall\, k \in \mathbb{Z}$.

An exceptional sequence $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ *is called strongly exceptional if the condition* (ES4) *holds, where*

$$\text{(ES4) } \mathrm{Hom}_\mathcal{T}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0\ \forall\, i, j \in [1, t],\ \forall\, k \in \mathbb{Z} - \{0\}.$$

We recall that strongly exceptional sequences appear very often in algebraic geometry and provides a non-commutative model for the study of algebraic varieties (see [7]).

Observe that any strongly exceptional sequence $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ of size t, in the triangulated category $\mathcal{T}$, is an example of a homological system in $\mathcal{T}$. Namely, the pair $(\mathcal{E}, \leq)$, for $\leq$ the natural order on $[1, t]$, is an $\mathcal{E}$-system in

$\mathcal{T}$. So, as an application of the developed theory of homological systems we get the following result.

Theorem 8.3. *Let $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ be a strongly exceptional sequence in an artin triangulated R-category $\mathcal{T}$, and let $E := \oplus_{i=1}^{t} \mathcal{E}_i$. Then, the following statements hold.*

(a) *The pair $(\mathfrak{F}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}))$ is an exact and Krull-Schmidt category.*

(b) *For the R-algebra $A := \mathrm{End}_{\mathcal{T}}(E)^{op}$, the derived functor*

$$R\mathrm{Hom}_{\mathcal{T}}(E, -) : \mathrm{D}^b(\mathfrak{F}(\mathcal{E})) \to \mathrm{D}^b(A)$$

is an equivalence as triangulated categories.

(c) *The pair $(A, \leq)$ is a quasi-hereditary algebra.*

Proof. By the condition (ES4), it follows that the triple $(\mathcal{E}, \mathcal{E}, \leq)$ is the $\mathcal{E}$-projective system associated to the $\mathcal{E}$-system $(\mathcal{E}, \leq)$. Thus, from 7.4 and the definition of strongly exceptional sequence the result follows. □

Remark 8.4. *Let $\mathcal{T} := \mathrm{D}^b(\mathrm{Sh}(X))$ be the bounded derived category of coherent sheaves on a smooth manifold X.*

(1) *In* [7, Theorem 6.2] *it is proven that, for any strongly exceptional sequence $\mathcal{E}$ in $\mathcal{T}$, such that $\mathcal{T}$ is generated by $\mathcal{E}$, the triangulated category $\mathcal{T}$ is equivalent to the bounded derived category $\mathrm{D}^b(A)$, where $A := \mathrm{End}_{\mathcal{T}}(E)^{op}$.*

(2) *Observe that, in 8.3, it is not assumed that $\mathcal{T}$ is generated by the strongly exceptional sequence $\mathcal{E}$.*

Now, we consider a hereditary abelian k-category $\mathcal{H}$, for some field k. By a result of Ringel (see [31, Theorem 1]), it follows that $\mathrm{Hom}_{\mathcal{T}}(\Theta, \Theta[-1]) = 0$ for any set Θ of indecomposable objects in $\mathcal{T} := \mathrm{D}^b(\mathcal{H})$. Hence, we get that any exceptional sequence $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ of size t, in the bounded derived category $\mathrm{D}^b(\mathcal{H})$, is an example of a homological system in $\mathrm{D}^b(\mathcal{H})$. Namely, the pair $(\mathcal{E}, \leq)$, for $\leq$ the natural order on $[1, t]$, is an $\mathcal{E}$-system in $\mathrm{D}^b(\mathcal{H})$. So, as an application of the developed theory of homological systems we get the following result.

Theorem 8.5. *Let $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t)$ be an exceptional sequence in the triangulated category $\mathcal{T} := \mathrm{D}^b(\mathcal{H})$. Then, the following statements hold true.*

(a) *The pair $(\mathfrak{F}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}))$ is an exact and Krull-Schmidt category.*

(b) *There exist a unique, up to isomorphism, families $\mathbf{Q}$ and $\mathbf{Y}$ of objects in $\mathcal{T}$, such that $(\mathcal{E}, \mathbf{Q}, \leq)$ is a $\mathcal{E}$-projective system and $(\mathcal{E}, \mathbf{Y}, \leq)$ is a $\mathcal{E}$-injective system.*

(c) *For the R-algebras $A := \mathrm{End}_{\mathcal{T}}(Q)^{op}$ and $B := \mathrm{End}_{\mathcal{T}}(Y)$, the derived functors*

$$R\mathrm{Hom}_{\mathcal{T}}(Q, -) : \mathrm{D}^b(\mathfrak{F}(\mathcal{E})) \to \mathrm{D}^b(A) \quad \textit{and} \quad R\mathrm{Hom}_{\mathcal{T}}(-, Y) : \mathrm{D}^b(\mathfrak{F}(\mathcal{E})) \to \mathrm{D}^b(B)$$

are equivalences as triangulated categories.

(d) *Both pairs $(A, \leq)$ and $(B, \leq^{op})$ are quasi-hereditary algebras, and moreover, the algebras A and B are derived equivalent.*

Proof. It follows from 7.4 and the definition of a exceptional sequence. $\square$

8.3. **A Θ-system which is not an exceptional sequence.** In what follows, we give an example of a Θ-system which is not a exceptional sequence and does not come from a stratifying system in a module category. To see that, we consider the hereditary path k-algebra $\Lambda := k(1 \longrightarrow 2 \longrightarrow 3)$ and the triangulated category $\mathcal{T} := \mathrm{D}^b(\Lambda)$. The Auslander-Reiten quiver of the bounded derived category $\mathrm{D}^b(\Lambda)$ can be seen in the Figure 1.

$$\begin{tikzcd}[column sep=tiny]
 & & & P_3[0] \arrow[dr] & & P_1[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_3[1] \arrow[dr] & & P_3[2] \arrow[dr] & & & \\
 & \cdots\cdots & P_2[0] \arrow[ur] \arrow[dr] & & \mathbf{I_2[0]} \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & & P_2[2] \arrow[ur] \arrow[dr] & & \mathbf{I_2[2]} \arrow[dr] & \cdots\cdots & \\
 & P_1[0] \arrow[ur] & & S_2[0] \arrow[ur] & & I_3[0] \arrow[ur] & & P_3[1] \arrow[ur] & & P_1[2] \arrow[ur] & & S_2[2] \arrow[ur] & & I_3[2] &
\end{tikzcd}$$

FIGURE 1. The bounded derived category $\mathrm{D}^b(\Lambda)$.

Consider the natural order $1 \leq 2 \leq 3$ and the set $\Theta := \{\Theta(1), \Theta(2), \Theta(3)\}$, of indecomposable objects in $\mathcal{T}$, where $\Theta(1) := I_2[0]$, $\Theta(2) := I_2[2]$ and $\Theta(3) := I_2[4]$. We assert that the pair $(\Theta, \leq)$ is a Θ-system of size 3 in the triangulated category $\mathcal{T}$. Indeed, by using Figure 1, it can be checked that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)) = 0$ for $j > i$. The condition $\mathrm{Hom}_{\mathcal{T}}(\Theta, \Theta[-1])$ follows from the fact that $\mathrm{mod}\,(\Lambda)$ is an abelian hereditary k-category. Finally, using that $\mathrm{Ext}^k_\Lambda(X, Y) \simeq \mathrm{Hom}_T(X[0], Y[k])$, it can be seen that $\mathrm{Hom}_{\mathcal{T}}(\Theta(j), \Theta(i)[1]) = 0$ for $j \geq i$. Therefore, the pair $(\Theta, \leq)$ is a Θ-system in $\mathcal{T}$.

Observe that $\mathrm{Hom}_{\mathcal{T}}(\Theta(3), \Theta(2)[2]) = \mathrm{Hom}_{\mathcal{T}}(\Theta(3), \Theta(3)) \neq 0$, and so Θ is not an exceptional sequence. Furthermore the pair $(\Theta, \leq)$ does not come from a stratifying system in the module category $\mathrm{mod}\,(\Lambda)$.

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REFERENCES

[1] I. Agoston, V. Dlab, E. Lukács. Stratified algebras. *Math. Rep. Acad. Sci. Canada* 20 (1) (1998) 22-28.
[2] I. Agoston, D. Happel, E. Lukács, L. Unger. Standardly Stratified algebras and tilting. *J. of algebra* 226 (1) (2000) 144-160.
[3] M. Auslander. Representation dimension of artin algebras. *Brandeis Univ.* (1973).
[4] M. Auslander, S. O. Smalo. Preprojective modules over Artin algebras. *J. of algebra* 66 (1980), 61-122.

[5] M. Barot, O. Mendoza. An explicit construction for the Happel functor. *Colloquium Math.*, No. 1, Vol. 104, 141-149, (2006).
[6] A. Beligiannis. Relative Homological algebra and Purity in triangulated categories. *Journal of algebra*, 227 (1), 268-361, (2000).
[7] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.* 53 (1989), 25-44 (in Russian); English transl.: Math. USSR-Izv. 34 (1990), 23-45.
[8] A. B. Buan. Subcategories of the derived category and cotilting complexes. *Colloq. Math.* 88 (2001), no. 1, 1-11.
[9] A. A. Beilinson, J. Bernstein, P. Deligne. Faisceaux perverse. *Asterisque* **100** (1982), 5-171.
[10] A. Beligiannis, I. Reiten. Homological and homotopical aspects of torsion theories. *Mem. Amer. Math. Soc.* 188 (2007) no. 883, viii+207 pp.
[11] Xiao-Wu Chen, Yu Ye, Pu Zhang. Algebras of derived dimension zero. *Comm. in algebra* 36, (2008), 1-10.
[12] Xiao-Wu Wu Chen. Extensions of covariantly finite subcategories. *Archiv der Mathematik* 93 (2009) 29-35.
[13] E. Cline, B.J. Parshall, L.L. Scott. Stratifying endomorphism algebras. *Memoirs of the AMS* 591, (1996).
[14] V. Dlab. Quasi-hereditary algebras revisited. *An. St. Univ. Ovidius Constanta* 4 (1996) 43-54.
[15] V. Dlab, C.M. Ringel. The Module Theoretical approach to Quasi-hereditary Algebras. *Repr. Theory and Related Topics, London Math. Soc. LNS* 168 (1992) 200-224.
[16] K. Erdmann, C. Sáenz. On standardly stratified algebras. *Comm. in algebra* 32 (2003) 3429-3446.
[17] O. Iyama, Y. Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay modules. *Inv. Math.* 172 (1), (2008), 117-168.
[18] S. I. Gelfand, Yu. I. Manin. Methods of homological algebra, *Springer Verlag*, Berlin.
[19] A. L. Gorodentsev and A. N. Rudakov. Exceptional vector bundles on projective spaces, *Duke Math. J.* 54 (1987), 115-130.
[20] B. Keller. Cluster tilted algebras are Gorenstein and stably Calaby-Yau. Adv. Math 211, 1, 123-151, (2007).
[21] B. Keller. Chain complexes and stable categories. *Manus. Math.* 67, 379-417, (1990).
[22] B. Keller. Derived categories and their uses. *Handbook of algebra*, Vol. 1, 671-701, *North-Holland, Amsterdam*, 1996.
[23] E.N. Marcos, O. Mendoza, C. Sáenz. Stratifying systems via relative simple modules. *J. of algebra* 280 (2004), 472-487.
[24] E.N. Marcos, O. Mendoza, C. Sáenz. Stratifying systems via relative projective modules. *Comm. in algebra* 33 (2005), 1559-1573.
[25] O. Mendoza, C. Sáenz, C. Xi. Homological systems in module categories over preordered sets. *Quart. J. Math.* 60 (2009), 75-103.
[26] H. Nakaoka. General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories. *Appl. Category structure.* 19 (6), (2011), 879-899.
[27] A. Neeman. The Derived Category of an Exact Category. *J. of algebra*, 135, 388-394, (1990).
[28] D. Pauksztello. Compact corigid objects in triangulated categories and co-t-structures. *Cent. Eur. J. Math.* **6** (1) (2008), 25-42.
[29] D. Quillen. Higher Algebraic K-theory I. *Springer LNM* 341, 85-147, (1973).
[30] C.M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.* 208 (1991) 209-233.

[31] C. M. Ringel. Hereditary triangulated categories. *Comp. Math.* (2005).
[32] A. N. Rudakov. Helices and vector bundles: seminaire Rudakov, volume 148 of London Mathematical Society Lecture Note Series. Cambridge University Press, 1990.
[33] C. Weibel. An Introduction to Homological Algebra. *Cambridge University Press*

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